

# The Complexity of Bayesian Networks Specified by Propositional and Relational Languages

Fabio G. Cozman  
Escola Politécnica  
Universidade de São Paulo

Denis D. Mauá  
Instituto de Matemática  
e Estatística  
Universidade de São Paulo

December 5, 2016

## Abstract

We examine the complexity of inference in Bayesian networks specified by logical languages. We consider representations that range from fragments of propositional logic to function-free first-order logic with equality; in doing so we cover a variety of plate models and of probabilistic relational models. We study the complexity of inferences when network, query and domain are the input (the *inferential* and the *combined* complexity), when the network is fixed and query and domain are the input (the *query/data* complexity), and when the network and query are fixed and the domain is the input (the *domain* complexity). We draw connections with probabilistic databases and liftability results, and obtain complexity classes that range from polynomial to exponential levels.

## 1 Introduction

A Bayesian network can represent any distribution over a given set of random variables [36, 71], and this flexibility has been used to great effect in many applications [109]. Indeed, Bayesian networks are routinely used to carry both deterministic and probabilistic assertions in a variety of knowledge representation tasks. Many of these tasks contain complex decision problems, with repetitive patterns of entities and relationships. Thus it is not surprising that practical concerns have led to modeling languages where Bayesian networks are specified using relations, logical variables, and quantifiers [49, 111]. Some of these languages enlarge Bayesian networks with plates [50, 85], while others resort to elements of database schema [47, 60]; some others mix probabilities with logic programming [106, 118] and even with functional programming [87, 89, 102]. The spectrum of tools that specify Bayesian networks by moving beyond propositional sentences is vast, and their applications are remarkable.

Yet most of the existing analysis on the complexity of inference with Bayesian networks focuses on a simplified setting where nodes of a network are associated with categorical variables and distributions are specified by flat tables containing probability values [115, 75]. This is certainly unsatisfying: as a point of comparison, consider the topic of *logical* inference, where much is known about the impact of specific constructs on computational complexity — suffice to mention the beautiful and detailed study of satisfiability in description logics [3].

In this paper we explore the complexity of inferences as dependent on the *language* that is used to specify the network. We adopt a simple specification strategy inspired by probabilistic programming [107] and by structural equation models [101]: A Bayesian network over binary variables is specified by a set of logical equivalences and a set of independent random variables. Using this simple scheme, we can parameterize computational complexity by the formal language that is allowed in the logical equivalences; we can move from sub-Boolean languages to relational ones, in the way producing languages that are similar in power to plate models [50] and to probabilistic relational models [74]. Note

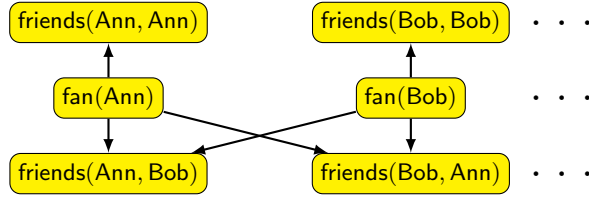


Figure 1: A Bayesian network with a repetitive pattern concerning friendship (only two students are shown; a larger network is obtained for a larger number of students).

that we follow a proven strategy adopted in logical formalisms: we focus on minimal sets of constructs (Boolean operators, quantifiers) that capture the essential connections between expressivity and complexity, and that can shed light on this connection for more sophisticated languages if needed. Our overall hope is to help with the design of knowledge representation formalisms, and in that setting it is important to understand the complexity introduced by language features, however costly those may be.

To illustrate the sort of specification we contemplate, consider a simple example that will be elaborated later. Suppose we have a population of students, and we denote by  $\text{fan}(\chi)$  the fact that student  $\chi$  is a fan of say a particular band. And we write  $\text{friends}(\chi, y)$  to indicate that  $\chi$  is a friend of  $y$ . Now consider a Bayesian network with a node  $\text{fan}(\chi)$  per student, and a node  $\text{friends}(\chi, y)$  per pair of students (see Figure 1). Suppose each node  $\text{fan}(\chi)$  is associated with the assessment  $\mathbb{P}(\text{fan}(\chi) = \text{true}) = 0.2$ . And finally suppose that a person is always a friend of herself, and two people are friends if they are fans of the band; that is, for each pair of students,  $\text{friends}(\chi, y)$  is associated with the formula

$$\text{friends}(\chi, y) \leftrightarrow (\chi = y) \vee (\text{fan}(\chi) \wedge \text{fan}(y)). \quad (1)$$

Now if we have data on some students, we may ask for the probability that some two students are friends, or the probability that a student is a fan. We may wish to consider more sophisticated formulas specifying friendship: how would the complexity of our inferences change, say, if we allowed quantifiers in our formula? Or if we allowed relations of arity higher than two? Such questions are the object of our discussion.

In this study, we distinguish a few concepts. *Inferential complexity* is the complexity when the network, the query and the domain are given as input. When the specification vocabulary is fixed, inference complexity is akin to *combined complexity* as employed in database theory. *Query complexity* is the complexity when the network is fixed and the input consists of query and domain. Query complexity has often been defined, in the context of probabilistic databases, as *data complexity* [123]. Finally, *domain complexity* is the complexity when network and query are fixed, and only the domain is given as input. Query and domain complexity are directly related respectively to *dqe-liftability* and *domain liftability*, concepts that have been used in lifted inference [9, 66]. We make connections with lifted inference and probabilistic databases whenever possible, and benefit from deep results originated from those topics. One of the contributions of this paper is a framework that can unify these varied research efforts with respect to the analysis of Bayesian networks. We show that many non-trivial complexity classes characterize the cost of inference as induced by various languages, and we make an effort to relate our investigation to various knowledge representation formalisms, from probabilistic description logics to plates to probabilistic relational models.

The paper is organized as follows. Section 2 reviews a few concepts concerning Bayesian networks and computational complexity. Our contributions start in Section 3, where we focus on propositional languages. In Section 4 we extend our framework to relational languages, and review relevant literature on probabilistic databases and lifted inference. In Sections 5 and 6 we study a variety of relational Bayesian network specifications. In Section 7 we connect these specifications to other schemes proposed in the literature. And in Section 8 we relate our results, mostly presented for decision problems, to

Valiant’s counting classes and their extensions. Section 9 summarizes our findings and proposes future work.

*All proofs are collected in A.*

## 2 A bit of notation and terminology

We denote by  $\mathbb{P}(A)$  the probability of event  $A$ . In this paper, every random variable  $X$  is a function from a finite sample space (usually a space with finitely many truth assignments or interpretations) to real numbers (usually to  $\{0, 1\}$ ). We refer to an event  $\{X = x\}$  as an *assignment*. Say that  $\{X = 1\}$  is a *positive* assignment, and  $\{X = 0\}$  is a *negative* assignment.

A *graph* consists of a set of *nodes* and a set of *edges* (an edge is a pair of nodes), and we focus on graphs that are directed and acyclic [71]. The parents of a node  $X$ , for a given graph, are denoted  $\text{pa}(X)$ . Suppose we have a directed acyclic graph  $\mathbb{G}$  such that each node is a random variable, and we also have a joint probability distribution  $\mathbb{P}$  over these random variables. Say that  $\mathbb{G}$  and  $\mathbb{P}$  satisfy the Markov condition iff each random variable  $X$  is independent of its nondescendants (in the graph) given its parents (in the graph).

A *Bayesian network* is a pair consisting of a directed acyclic graph  $\mathbb{G}$  whose nodes are random variables and a joint probability distribution  $\mathbb{P}$  over all variables in the graph, such that  $\mathbb{G}$  and  $\mathbb{P}$  satisfy the Markov condition [92]. For a collection of measurable sets  $A_1, \dots, A_n$ , we then have

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \prod_{i=1}^n \mathbb{P}\left(X_i \in A_i \mid \text{pa}(X_i) \in \bigcap_{j: X_j \in \text{pa}(X_i)} A_j\right)$$

whenever the conditional probabilities exist. If all random variables are discrete, then one can specify “local” conditional probabilities  $\mathbb{P}(X_i = x_i \mid \text{pa}(X_i) = \pi_i)$ , and the joint probability distribution is necessarily the product of these local probabilities:

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n \mathbb{P}(X_i = x_i \mid \text{pa}(X_i) = \pi_i), \quad (2)$$

where  $\pi_i$  is the projection of  $\{x_1, \dots, x_n\}$  on  $\text{pa}(X_i)$ , with the understanding that  $\mathbb{P}(X_i = x_i \mid \text{pa}(X_i) = \pi_i)$  stands for  $\mathbb{P}(X_i = x_i)$  whenever  $X_i$  has not parents.

In this paper we only deal with finite objects, so we can assume that a Bayesian network is fully specified by a finite graph and a local conditional probability distribution per random variable (the local distribution associated with random variable  $X$  specifies the probability of  $X$  given the parents of  $X$ ). Often probability values are given in tables (referred to as *conditional probability tables*). Depending on how these tables are encoded, the directed acyclic graph may be redundant; that is, all the information to reconstruct the graph and the joint distribution is already in the tables. In fact we rarely mention the graph  $\mathbb{G}$  in our results; however graphs are visually useful and we often resort to drawing them in our examples.

A basic computational problem for Bayesian networks is: Given a Bayesian network  $\mathbb{B}$ , a set of assignments  $\mathbf{Q}$  and a set of assignments  $\mathbf{E}$ , determine whether  $\mathbb{P}(\mathbf{Q} \mid \mathbf{E}) > \gamma$  for some rational number  $\gamma$ . We assume that every probability value is specified as a rational number. Thus,  $\mathbb{P}(\mathbf{Q} \mid \mathbf{E}) = \mathbb{P}(\mathbf{Q}, \mathbf{E}) / \mathbb{P}(\mathbf{E})$  is a rational number, as  $\mathbb{P}(\mathbf{Q}, \mathbf{E})$  and  $\mathbb{P}(\mathbf{E})$  are computed by summing through products given by Expression (2).

We adopt basic terminology and notation from computational complexity [98]. A *language* is a set of strings. A language defines a *decision problem*; that is, the problem of deciding whether an input string is in the language. A *complexity class* is a set of languages; we use well-known complexity classes  $\mathbf{P}$ ,  $\mathbf{NP}$ ,  $\mathbf{PSPACE}$ ,  $\mathbf{EXP}$ ,  $\mathbf{ETIME}$ ,  $\mathbf{NETIME}$ . The complexity class  $\mathbf{PP}$  consists of those languages  $\mathcal{L}$  that satisfy the following property: there is a polynomial time nondeterministic Turing machine  $M$  such that  $\ell \in \mathcal{L}$  iff more than half of the computations of  $M$  on input  $\ell$  end up accepting. Analogously, we

have PEXP, consisting of those languages  $\mathcal{L}$  with the following property: there is an exponential time nondeterministic Turing machine  $M$  such that  $\ell \in \mathcal{L}$  iff half of the computations of  $M$  on input  $\ell$  end up accepting [15].

To proceed, we need to define oracles and related complexity classes. An oracle Turing machine  $M^{\mathcal{L}}$ , where  $\mathcal{L}$  is a language, is a Turing machine with additional tapes, such that it can write a string  $\ell$  to a tape and obtain from the oracle, in unit time, the decision as to whether  $\ell \in \mathcal{L}$  or not. If a class of languages/functions  $\mathbf{A}$  is defined by a set of Turing machines  $\mathcal{M}$  (that is, the languages/functions are decided/computed by these machines), then define  $\mathbf{A}^{\mathcal{L}}$  to be the set of languages/functions that are decided/computed by  $\{M^{\mathcal{L}} : M \in \mathcal{M}\}$ . For a function  $f$ , an oracle Turing machine  $M^f$  can be similarly defined, and for any class  $\mathbf{A}$  we have  $\mathbf{A}^f$ . If  $\mathbf{A}$  and  $\mathbf{B}$  are classes of languages/functions,  $\mathbf{A}^{\mathbf{B}} = \bigcup_{x \in \mathbf{B}} \mathbf{A}^x$ . For instance, the *polynomial hierarchy* consists of classes  $\Sigma_i^{\mathbf{P}} = \mathbf{NP}^{\Sigma_{i-1}^{\mathbf{P}}}$  and  $\Pi_i^{\mathbf{P}} = \mathbf{co}\Sigma_i^{\mathbf{P}}$ , with  $\Sigma_0^{\mathbf{P}} = \mathbf{P}$  (and PH is the union  $\bigcup_i \Pi_i^{\mathbf{P}} = \bigcup_i \Sigma_i^{\mathbf{P}}$ ).

We examine Valiant's approach to counting problems in Section 8; for now suffice to say that  $\#\mathbf{P}$  is the class of functions such that  $f \in \#\mathbf{P}$  iff  $f(\ell)$  is the number of computation paths that accept  $\ell$  for some polynomial time nondeterministic Turing machine [130]. It is as if we had a special machine, called by Valiant a *counting* Turing machine, that on input  $\ell$  prints on a special tape the number of computations that accept  $\ell$ .

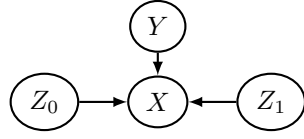
We will also use the class  $\mathbf{PP}_1$ , defined as the set of languages in  $\mathbf{PP}$  that have a single symbol as input vocabulary. We can take this symbol to be 1, so the input is just a sequence of 1s (one can interpret this input as a non-negative integer written in unary notation). This is the counterpart of Valiant's class  $\#\mathbf{P}_1$  that consists of the functions in  $\#\mathbf{P}$  that have a single symbol as input vocabulary [131].

We focus on many-one reductions: such a reduction from  $\mathcal{L}$  to  $\mathcal{L}'$  is a polynomial time algorithm that takes the input to decision problem  $\mathcal{L}$  and transforms it into the input to decision problem  $\mathcal{L}'$  such that  $\mathcal{L}'$  has the same output as  $\mathcal{L}$ . A *Turing reduction* from  $\mathcal{L}$  to  $\mathcal{L}'$  is a polynomial time algorithm that decides  $\mathcal{L}$  using  $\mathcal{L}'$  as an oracle. For a complexity class  $\mathbf{C}$ , a decision problem  $\mathcal{L}$  is  $\mathbf{C}$ -hard with respect to many-one reductions if each decision problem in  $\mathbf{C}$  can be reduced to  $\mathcal{L}$  with many-one reductions. A decision problem is then  $\mathbf{C}$ -complete with respect to many-one reductions if it is in  $\mathbf{C}$  and it is  $\mathbf{C}$ -hard with respect to many-one reductions. Similar definitions of hardness and completeness are obtained when "many-one reductions" are replaced by "Turing reductions".

An important  $\mathbf{PP}$ -complete (with respect to many-one reductions) decision problem is MAJSAT: the input is a propositional sentence  $\phi$  and the decision is whether or not the majority of assignments to the propositions in  $\phi$  make  $\phi$  true [51]. Another  $\mathbf{PP}$ -complete problem (with respect to many-one reductions) is deciding whether the number of satisfying assignments for  $\phi$  is larger than an input integer  $k$  [119]; in fact this problem is still  $\mathbf{PP}$ -complete with respect to many-one reductions even if  $\phi$  is monotone [54]. Recall that a sentence is *monotone* if it has no negation.

A formula is in  $k$ CNF iff it is in Conjunctive Normal Form with  $k$  literals per clause (if there is no restriction on  $k$ , we just write CNF). MAJSAT is  $\mathbf{PP}$ -complete with respect to many-one reductions even if the input is restricted to be in CNF; however, it is not known whether MAJSAT is still  $\mathbf{PP}$ -complete with respect to many-one reductions if the sentence  $\phi$  is in 3CNF. Hence we will resort in proofs to a slightly different decision problem, following results by Bailey et al. [7]. The problem  $\#3\text{SAT}(>)$  gets as input a propositional sentence  $\phi$  in 3CNF and an integer  $k$ , and the decision is whether  $\#\phi > k$ ; we use, here and later in proofs,  $\#\phi$  to denote the number of satisfying assignments for a formula  $\phi$ . We will also use, in the proof of Theorem 2, the following decision problem. Say that an assignment to the propositions in a sentence in CNF *respects* the 1-in-3 rule if at most one literal per clause is assigned true. Denote by  $\#(1\text{-in-}3)\phi$  the number of satisfying assignments for  $\phi$  that also respects the 1-in-3 rule. The decision problem  $\#(1\text{-in-}3)\text{SAT}(>)$  gets as input a propositional sentence  $\phi$  in 3CNF and an integer  $k$ , and decides whether  $\#(1\text{-in-}3)\phi > k$ . We have:

**Proposition 1.** *Both  $\#3\text{SAT}(>)$  and  $\#(1\text{-in-}3)\text{SAT}(>)$  are  $\mathbf{PP}$ -complete with respect to many-one reductions.*



$$\begin{aligned}\mathbb{P}(Y = 1) &= 1/3, \\ \mathbb{P}(Z_0 = 1) &= 1/5, \quad \mathbb{P}(Z_1 = 1) = 7/10, \\ X &\equiv (Y \wedge Z_1) \vee (\neg Y \wedge Z_0).\end{aligned}$$

Figure 2: A Bayesian network specified with logical equivalences and unconditional probabilistic assessments.

### 3 Propositional languages: Inferential and query complexity

In this section we focus on propositional languages, so as to present our proposed framework in the most accessible manner. Recall that we wish to parameterize the complexity of inferences by the language used in specifying local distributions.

#### 3.1 A specification framework

We are interested in specifying Bayesian networks over binary variables  $X_1, \dots, X_n$ , where each random variable  $X_i$  is the indicator function of a proposition  $A_i$ . That is, consider the space  $\Omega$  consisting of all truth assignments for these variables (there are  $2^n$  such truth assignments); then  $X_i$  yields 1 for a truth assignment that satisfies  $A_i$ , and  $X_i$  yields 0 for a truth assignment that does not satisfy  $A_i$ .

*We will often use the same letter to refer to a proposition and the random variable that is the indicator function of the proposition.*

We adopt a specification strategy that moves away from tables of probability values, and that is inspired by probabilistic programming [103, 118] and by structural models [100]. A *Bayesian network specification* associates with each proposition  $X_i$  either

- a logical equivalence  $X_i \leftrightarrow \ell_i$ , or
- a probabilistic assessment  $\mathbb{P}(X_i = 1) = \alpha$ ,

where  $\ell_i$  is a formula in a propositional language  $\mathcal{L}$ , such that the only extralogical symbols in  $\ell_i$  are propositions in  $\{X_1, \dots, X_n\}$ , and  $\alpha$  is a rational number in the interval  $[0, 1]$ .

We refer to each logical equivalence  $X_i \leftrightarrow \ell_i$  as a *definition axiom*, borrowing terminology from description logics [3]. We refer to  $\ell_i$  as the *body* of the definition axiom. In order to avoid confusion between the leftmost symbol  $\leftrightarrow$  and possible logical equivalences within  $\ell_i$ , we write a definition axiom as in description logics:

$$X_i \equiv \ell_i,$$

and we emphasize that  $\equiv$  is just syntactic sugar for logical equivalence  $\leftrightarrow$ .

A Bayesian network specification induces a directed graph where the nodes are the random variables  $X_1, \dots, X_n$ , and  $X_j$  is a parent of  $X_i$  if and only if the definition axiom for  $X_i$  contains  $X_j$ . If this graph is acyclic, as we assume in this paper, then the Bayesian network specification does define a Bayesian network.

Figure 2 depicts a Bayesian network specified this way.

Note that we avoid direct assessments of conditional probability, because one can essentially create negation through  $\mathbb{P}(X = 1|Y = 1) = \mathbb{P}(X = 0|Y = 0) = 0$ . In our framework, the use of negation is a decision about the language. We will see that negation does make a difference when complexity is analyzed.

Any distribution over binary variables given by a Bayesian network can be equivalently defined using definition axioms, as long as definitions are allowed to contain negation and conjunction (and then disjunction is syntactic sugar). To see that, consider a conditional distribution for  $X$  given  $Y_1$

and  $Y_2$ ; we can specify this distribution using the definition axiom

$$X \equiv (\neg Y_1 \wedge \neg Y_2 \wedge Z_{00}) \vee (\neg Y_1 \wedge Y_2 \wedge Z_{01}) \vee (Y_1 \wedge \neg Y_2 \wedge Z_{10}) \vee (Y_1 \wedge Y_2 \wedge Z_{11}),$$

where  $Z_{ab}$  are fresh binary variables (that do not appear anywhere else), associated with assessments  $\mathbb{P}(Z_{ab} = 1) = \mathbb{P}(X = 1 | Y_1 = a, Y_2 = b)$ . This sort of encoding can be extended to any set  $Y_1, \dots, Y_m$  of parents, demanding the same space as the corresponding conditional probability table.

**Example 1.** Consider a simple Bayesian network with random variables  $X$  and  $Y$ , where  $Y$  is the sole parent of  $X$ , and where:

$$\mathbb{P}(Y = 1) = 1/3, \quad \mathbb{P}(X = 1 | Y = 0) = 1/5, \quad \mathbb{P}(X = 1 | Y = 1) = 7/10.$$

Then Figure 2 presents an equivalent specification for this network, in the sense that both specifications have the same marginal distribution over  $(X, Y)$ .  $\square$

Note that definition axioms can exploit structures that conditional probability tables cannot; for instance, to create a Noisy-Or gate [99], we simply say that  $X \equiv (Y_1 \wedge W_1) \vee (Y_2 \wedge W_2)$ , where  $W_1$  and  $W_2$  are inhibitor variables.

### 3.2 The complexity of propositional languages

Now consider a language  $\text{INF}[\mathcal{L}]$  that consists of the strings  $(\mathbb{B}, \mathbf{Q}, \mathbf{E}, \gamma)$  for which  $\mathbb{P}(\mathbf{Q} | \mathbf{E}) > \gamma$ , where

- $\mathbb{P}$  is the distribution encoded by a Bayesian network specification  $\mathbb{B}$  with definition axioms whose bodies are formulas in  $\mathcal{L}$ ,
- $\mathbf{Q}$  and  $\mathbf{E}$  are sets of assignments (the *query*),
- and  $\gamma$  is a rational number in  $[0, 1]$ .

For instance, denote by  $\text{Prop}(\wedge, \neg)$  the language of propositional formulas containing conjunction and negation. Then  $\text{INF}[\text{Prop}(\wedge, \neg)]$  is the language that decides the probability of a query for networks specified with definition axioms containing conjunction and negation. As every Bayesian network over binary variables can be specified with such definition axioms,  $\text{INF}[\text{Prop}(\wedge, \neg)]$  is in fact a PP-complete language [36, Theorems 11.3 and 11.5].

There is obvious interest in finding simple languages  $\mathcal{L}$  such that deciding  $\text{INF}[\mathcal{L}]$  is a tractable problem, so as to facilitate elicitation, decision-making and learning [34, 40, 64, 108, 116]. And there are indeed propositional languages that generate tractable Bayesian networks: for instance, it is well known that *Noisy-Or* networks display polynomial inference when the query consists of negative assignments [59]. Recall that a Noisy-Or network has a *bipartite* graph with edges pointing from nodes in one set to nodes in the other set, and the latter nodes are associated with Noisy-Or gates.

One might think that tractability can only be attained by imposing some structural conditions on graphs, given results that connect complexity and graph properties [76]. However, it is possible to attain tractability without restrictions on graph topology. Consider the following result, where we use  $\text{Prop}(\nu)$  to indicate a propositional language with operators restricted to the ones in the list  $\nu$ :

**Theorem 1.**  $\text{INF}[\text{Prop}(\wedge)]$  is in to P when the query  $(\mathbf{Q}, \mathbf{E})$  contains only positive assignments, and  $\text{INF}[\text{Prop}(\vee)]$  is in to P when the query contains only negative assignments.

As the proof of this result shows (in A), only polynomial effort is needed to compute probabilities for positive queries in networks specified with  $\text{Prop}(\wedge)$ , *even* if one allows root nodes to be negated (that is, the variables that appear in probabilistic assessments can appear negated in the body of definition axioms).

Alas, even small movements away from the conditions in Theorem 1 takes us to PP-completeness:

**Theorem 2.**  $\text{INF}[\text{Prop}(\wedge)]$  and  $\text{INF}[\text{Prop}(\vee)]$  are PP-complete with respect to many-one reductions.

Proofs for these results are somewhat delicate due to the restriction to many-one reductions. In A we show that much simpler proofs for PP-completeness of  $\text{INF}(\text{Prop}(\wedge))$  and  $\text{INF}(\text{Prop}(\vee))$  are possible if one uses Turing reductions. A Turing reduction gives some valuable information: if a problem is PP-complete with Turing reductions, then it is unlikely to be polynomial (for if it were polynomial, then  $\text{P}^{\text{PP}}$  would equal P, a highly unlikely result given current assumptions in complexity theory [127]). However, Turing reductions tend to blur some significant distinctions. For instance, for Turing reductions it does not matter whether  $\mathbf{Q}$  is a singleton or not: one can ask for  $\mathbb{P}(Q_1|\mathbf{E}_1)$ ,  $\mathbb{P}(Q_2|\mathbf{E}_2)$ , and so on, and then obtain  $\mathbb{P}(Q_1, Q_2, \dots|\mathbf{E})$  as the product of the intermediate computations. However, it may be the case that for some languages such a distinction concerning  $\mathbf{Q}$  matters. Hence many-one reductions yield stronger results, so we emphasize them throughout this paper.

One might try to concoct additional languages by using specific logical forms in the literature [37]. We leave this to future work; instead of pursuing various possible sub-Boolean languages, we wish to quickly examine the *query complexity* of Bayesian networks, and then move to relational languages in Section 4.

### 3.3 Query complexity

We have so far considered that the input is a string encoding a Bayesian network specification  $\mathbb{B}$ , a query  $(\mathbf{Q}, \mathbf{E})$ , and a rational number  $\gamma$ . However in practice one may face a situation where the Bayesian network is fixed, and the input is a string consisting of the pair  $(\mathbf{Q}, \mathbf{E})$  and a rational number  $\gamma$ ; the goal is to determine whether  $\mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma$  with respect to the fixed Bayesian network.

Denote by  $\text{QINF}[\mathbb{B}]$ , where  $\mathbb{B}$  is a Bayesian network specification, the language consisting of each string  $(\mathbf{Q}, \mathbf{E}, \gamma)$  for which  $\mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma$  with respect to  $\mathbb{B}$ . And denote by  $\text{QINF}[\mathcal{L}]$  the set of languages  $\text{QINF}[\mathbb{B}]$  where  $\mathbb{B}$  is a Bayesian network specification with definition axioms whose bodies are formulas in  $\mathcal{L}$ .

**Definition 1.** Let  $\mathcal{L}$  be a propositional language and  $\mathbf{C}$  be a complexity class. The query complexity of  $\mathcal{L}$  is  $\mathbf{C}$  if and only if every language in  $\text{QINF}[\mathcal{L}]$  is in  $\mathbf{C}$ .

The fact that query complexity may differ from inferential complexity was initially raised by Darwiche and Provan [34], and has led to a number of techniques emphasizing compilation of a fixed Bayesian network [23, 35]. Indeed the expression “query complexity” seems to have been coined by Darwiche [36, Section 6.9], without the formal definition presented here.

The original work by Darwiche and Provan [34] shows how to transform a fixed Bayesian network into a *Query-DAG* such that  $\mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma$  can be decided in linear time. That is:

**Theorem 3** (Darwiche and Provan [34]).  $\text{QINF}[\text{Prop}(\wedge, \neg)]$  is in P.

Results on query complexity become more interesting when we move to relational languages.

## 4 Relational Languages: Inferential, query, and domain complexity

In this section we extend our specification framework so as to analyze the complexity of relational languages. Such languages have been used in a variety of applications with repetitive entities and relationships [49, 111].

### 4.1 Relational Bayesian network specifications

We start by blending some terminology and notation by Poole [105] and by Milch et al. [90].



Figure 3: Representing dependences amongst relations in Example 2.

A *parameterized random variable*, abbreviated *parvariable*, is a function that yields, for each combination of its input parameters, a random variable. For instance, parvariable  $X$  yields a random variable  $X(\chi)$  for each  $\chi$ . In what follows, parvariables and their parameters will correspond to relations and their logical variables.

We use a *vocabulary* consisting of names of relations. Every relation  $X$  is associated with a non-negative integer called its *arity*. We also use logical variables; a logical variable is referred to as a *logvar*. A vector of logvars  $[\chi_1, \dots, \chi_k]$  is denoted  $\vec{\chi}$ ; then  $X(\vec{\chi})$  is an *atom*. A *domain* is a set; in this paper every domain is finite. When the logvars in an atom are replaced by elements of the domain, we obtain  $X(a_1, \dots, a_k)$ , a *ground atom*, often referred to as a *grounding* of relation  $X$ . An *interpretation*  $\mathbb{I}$  is a function that assigns to each relation  $X$  of arity  $k$  a relation on  $\mathcal{D}^k$ . An interpretation can be viewed as a function that assigns **true** or **false** to each grounding  $X(\vec{a})$ , where  $\vec{a}$  is a tuple of elements of the domain. Typically in logical languages there is a distinction between *constants* and elements of a domain, but we avoid constants altogether in our discussion (as argued by Bacchus, if constants are used within a probabilistic logic, some sort of additional *rigidity* assumption must be used [4]).

Given a domain  $\mathcal{D}$ , we can associate with each grounding  $X(\vec{a})$  a random variable  $\hat{X}(\vec{a})$  over the set of all possible interpretations, such that  $\hat{X}(\vec{a})(\mathbb{I}) = 1$  if interpretation  $\mathbb{I}$  assigns **true** to  $X(\vec{a})$ , and  $\hat{X}(\vec{a})(\mathbb{I}) = 0$  otherwise. Similarly, we can associate with a relation  $X$  a parvariable  $\hat{X}$  that yields, once a domain is given, a random variable  $\hat{X}(\vec{a})$  for each grounding  $X(\vec{a})$ . To simplify matters, we use the same symbol for a grounding  $X(\vec{a})$  and its associated random variable  $\hat{X}(\vec{a})$ , much as we did with propositions and their associated random variables. Similarly, we use the same symbol for a relation  $X$  and its associated parvariable  $\hat{X}$ . We can then write down logical formulas over relations/parvariables, and we can assess probabilities for relations/parvariables. The next example clarifies the dual use of symbols for relations/parvariables.

**Example 2.** Consider a model of friendship built on top of the example in Section 1. Two people are friends if they are both fans of the same band, or if they are linked in some other unmodeled way, and a person is always a friend of herself. Take relations *friends*, *fan*, and *linked*. Given a domain, say  $\mathcal{D} = \{a, b\}$ , we have the grounding *friends*( $a, b$ ), whose intended interpretation is that  $a$  and  $b$  are friends; we take friendship to be asymmetric so *friends*( $a, b$ ) may hold while *friends*( $b, a$ ) may not hold. We also have groundings *fan*( $a$ ), *linked*( $b, a$ ), and so on. Each one of these groundings corresponds to a random variable that yields 1 or 0 when the grounding is respectively **true** or **false** is an interpretation.

The stated facts about friendship might be encoded by an extended version of Formula (1), written here with the symbol  $\equiv$  standing for logical equivalence:

$$\text{friends}(\chi, y) \equiv (\chi = y) \vee (\text{fan}(\chi) \wedge \text{fan}(y)) \vee \text{linked}(\chi, y). \quad (3)$$

We can draw a directed graph indicating the dependence of *friends* on the other relations, as in Figure 3. Suppose we believe 0.2 is the probability that an element of the domain is a fan, and 0.1 is the probability that two people are linked for some other reason. To express these assessments we might write

$$\mathbb{P}(\text{fan}(\chi) = 1) = 0.2 \quad \text{and} \quad \mathbb{P}(\text{linked}(\chi, y) = 1) = 0.1, \quad (4)$$

with implicit outer universal quantification.  $\square$

Given a formula and a domain, we can produce all groundings of the formula by replacing its logvars by elements of the domain in every possible way (as usual when grounding first-order formulas). We can similarly ground probabilistic assessments by grounding the affected relations.



**Example 3.** In Example 2, we can produce the following groundings from domain  $\mathcal{D} = \{a, b\}$  and Formula (3):

$$\begin{aligned} \text{friends}(a, a) &\equiv (a = a) \vee (\text{fan}(a) \wedge \text{fan}(a)) \vee \text{linked}(a, a), \\ \text{friends}(a, b) &\equiv (a = b) \vee (\text{fan}(a) \wedge \text{fan}(b)) \vee \text{linked}(a, b), \\ \text{friends}(b, a) &\equiv (b = a) \vee (\text{fan}(b) \wedge \text{fan}(a)) \vee \text{linked}(b, a), \\ \text{friends}(b, b) &\equiv (b = b) \vee (\text{fan}(b) \wedge \text{fan}(b)) \vee \text{linked}(b, b), \end{aligned}$$

Similarly, we obtain:

$$\begin{aligned} \mathbb{P}(\text{fan}(a) = 1) &= 0.2, & \mathbb{P}(\text{fan}(b) = 1) &= 0.2, \\ \mathbb{P}(\text{linked}(a, a) = 1) &= 0.1, & \mathbb{P}(\text{linked}(a, b) = 1) &= 0.1, \\ \mathbb{P}(\text{linked}(b, a) = 1) &= 0.1, & \mathbb{P}(\text{linked}(b, b) = 1) &= 0.1, \end{aligned}$$

by grounding assessments in Expression (4).  $\square$

In short: we wish to extend our propositional framework by specifying Bayesian networks using both parameterized probabilistic assessments and first-order definitions. So, suppose we have a finite set of parvariables, each one of them corresponding to a relation in a vocabulary. A *relational Bayesian network specification* associates, with each parvariable  $X_i$ , either

- a *definition axiom*  $X_i(\vec{x}) \equiv \ell_i(\vec{x}, Y_1, \dots, Y_m)$ , or
- a *probabilistic assessment*  $\mathbb{P}(X(\vec{x}) = 1) = \alpha$ ,

where

- $\ell_i$  is a well-formed formula in a language  $\mathcal{L}$ , containing relations  $Y_1, \dots, Y_m$  and free logvars  $\vec{x}$  (and possibly additional logvars bound to quantifiers),
- and  $\alpha$  is a rational number in  $[0, 1]$ .

The formula  $\ell_i$  is the *body* of the corresponding definition axiom. The parvariables that appear in  $\ell_i$  are the *parents* of parvariable  $X_i$ , and are denoted by  $\text{pa}(X_i)$ . Clearly the definition axioms induce a directed graph where the nodes are the parvariables and the parents of a parvariable (in the graph) are exactly  $\text{pa}(X_i)$ . This is the *parvariable graph* of the relational Bayesian network specification (this sort of graph is called a *template dependency graph* by Koller and Friedman [71, Definition 6.13]). For instance, Figure 3 depicts the parvariable graph for Example 2.

When the parvariable graph of a relational Bayesian network specification is acyclic, we say the specification itself is acyclic. *In this paper we assume that relational Bayesian network specifications are acyclic*, and we do not even mention this anymore.

The *grounding* of a relational Bayesian network specification  $\mathbb{S}$  on a domain  $\mathcal{D}$  is defined as follows. First, produce all groundings of all definition axioms. Then, for each parameterized probabilistic assessment  $\mathbb{P}(X(\vec{x}) = 1) = \alpha$ , produce its ground probabilistic assessments

$$\mathbb{P}(X(\vec{a}_1) = 1) = \alpha, \quad \mathbb{P}(X(\vec{a}_2) = 1) = \alpha, \quad \text{and so on,}$$

for all appropriate tuples  $\vec{a}_j$  built from the domain. The grounded relations, definitions and assessments specify a propositional Bayesian network that is then the semantics of  $\mathbb{S}$  with respect to domain  $\mathcal{D}$ .

**Example 4.** Consider Example 2. For a domain  $\{a, b\}$ , the relational Bayesian network specification given by Expressions (3) and (4) is grounded into the sentences and assessments in Example 3. By repeating this process for a larger domain  $\{a, b, c\}$ , we obtain a larger Bayesian network whose graph is depicted in Figure 4.  $\square$

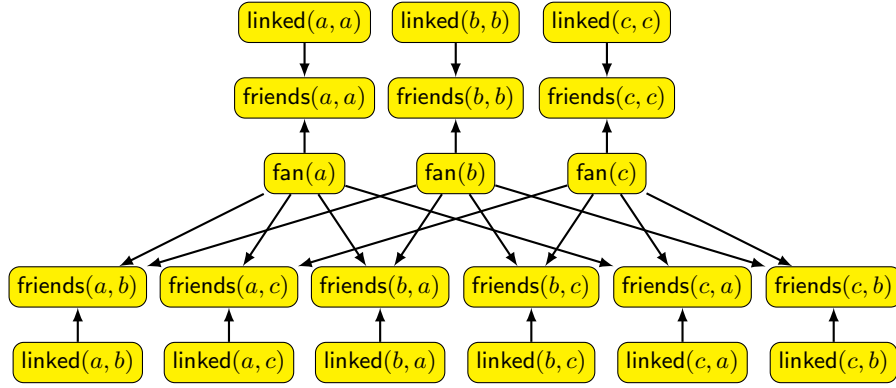


Figure 4: The grounding (on domain  $\{a, b, c\}$ ) of the relational Bayesian network specification in Example 2.

Note that logical inference might be used to simplify grounded definitions; for instance, in the previous example, one might note that  $\text{friends}(a, a)$  is simply true. Note also that the grounding of an formula with quantifiers turns, as usual, an existential quantifier into a disjunction, and a universal quantifier into a conjunction. Consider the example:

**Example 5.** Take the following relational Bayesian network specification (with no particular meaning, just to illustrate a few possibilities):

$$\begin{aligned} \mathbb{P}(X_1(\chi) = 1) &= 2/3, & \mathbb{P}(X_2(\chi) = 1) &= 1/10, \\ \mathbb{P}(X_3(\chi) = 1) &= 4/5, & \mathbb{P}(X_4(\chi, y) = 1) &= 1/2, \\ X_5(\chi) &\equiv \exists y : \forall z : \neg X_1(\chi) \vee X_2(y) \vee X_3(z), \\ X_6(\chi) &\equiv X_5(\chi) \wedge \exists y : X_4(\chi, y) \wedge X_1(y), \end{aligned}$$

Take a domain  $\mathcal{D} = \{1, 2\}$ ; the grounded definition of  $X_5(1)$  is

$$\begin{aligned} X_5(1) \equiv & ((\neg X_1(1) \vee X_2(1) \vee X_3(1)) \wedge (\neg X_1(1) \vee X_2(1) \vee X_3(2))) \vee \\ & ((\neg X_1(1) \vee X_2(2) \vee X_3(1)) \wedge (\neg X_1(1) \vee X_2(2) \vee X_3(2))). \end{aligned}$$

Figure 5 depicts the parvariable graph and the grounding of this relational Bayesian network specification.  $\square$

In order to study complexity questions we must decide how to encode any given domain. Note that there is no need to find special names for the elements of the domain, so we take that the domain is always the set of numbers  $\{1, 2, \dots, N\}$ . Now if this list is explicitly given as input, then the size of the input is of order  $N$ . However, if only the number  $N$  is given as input, then the size of the input is either of order  $N$  when  $N$  is encoded in unary notation, or of order  $\log N$  when  $N$  is encoded in binary notation. The distinction between unary and binary notation for input numbers is often used in description logics [3].

The conceptual difference between unary and binary encodings of domain size can be captured by the following analogy. Suppose we are interested in the inhabitants of a city: the probabilities that they study, that they marry, that they vote, and so on. Suppose the behavior of these inhabitants is modeled by a relational Bayesian network specification, and we observe evidence on a few people. If we then take our input  $N$  to be in unary notation, we are implicitly assuming that we have a directory, say a mailing list, with the names of all inhabitants; even if we do not care about their specific names, each one of them exists concretely in our modeled reality. But if we take our input  $N$  to be in binary notation, we are just focusing on the impact of city size on probabilities, without any regard for the actual inhabitants; we may say that  $N$  is a thousand, or maybe fifty million (and perhaps neither of these numbers is remotely accurate).

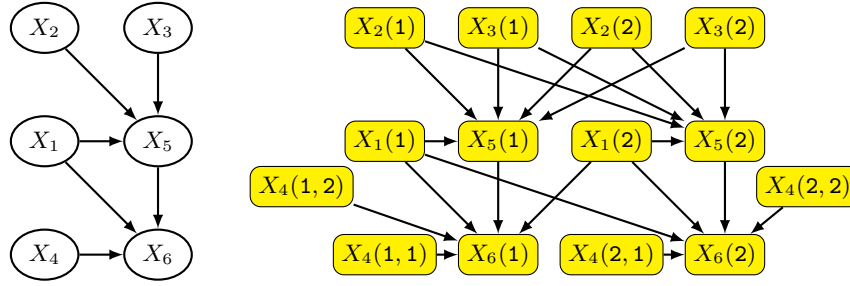


Figure 5: The parvariable graph of the relational Bayesian network specification in Example 5, and its grounding on domain  $\mathcal{D} = \{1, 2\}$ .

## 4.2 Inferential, combined, query and domain complexity

To repeat, we are interested in the relationship between the language  $\mathcal{L}$  that is employed in the body of definition axioms and the complexity of inferences. While in the propositional setting we distinguished between inferential and query complexity, here we have an additional distinction to make. Consider the following definitions, where  $\mathbb{S}$  is a relational Bayesian network specification,  $N$  is the domain size,  $\mathbf{Q}$  and  $\mathbf{E}$  are sets of assignments for ground atoms,  $\gamma$  is a rational number in  $[0, 1]$ , and  $\mathbf{C}$  is a complexity class:

**Definition 2.** Denote by  $\text{INF}[\mathcal{L}]$  the language consisting of strings  $(\mathbb{S}, N, \mathbf{Q}, \mathbf{E}, \gamma)$  for which  $\mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma$  with respect to the grounding of  $\mathbb{S}$  on domain of size  $N$ , where  $\mathbb{S}$  contains definition axioms whose bodies are formulas in  $\mathcal{L}$ . The inferential complexity of  $\mathcal{L}$  is  $\mathbf{C}$  iff  $\text{INF}[\mathcal{L}]$  is in  $\mathbf{C}$ ; moreover, the inferential complexity is  $\mathbf{C}$ -hard with respect to a reduction iff  $\text{INF}[\mathcal{L}]$  is  $\mathbf{C}$ -hard with respect to the reduction, and it is  $\mathbf{C}$ -complete with respect to a reduction iff it is in  $\mathbf{C}$  and it is  $\mathbf{C}$ -hard with respect to the reduction.

**Definition 3.** Denote by  $\text{QINF}[\mathbb{S}]$  the language consisting of strings  $(N, \mathbf{Q}, \mathbf{E}, \gamma)$  for which  $\mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma$  with respect to the grounding of  $\mathbb{S}$  on domain of size  $N$ . Denote by  $\text{QINF}[\mathcal{L}]$  the set of languages  $\text{QINF}[\mathbb{S}]$  for  $\mathbb{S}$  where the bodies of definition axioms in  $\mathbb{S}$  are formulas in  $\mathcal{L}$ . The query complexity of  $\mathcal{L}$  is in  $\mathbf{C}$  iff every language in  $\text{QINF}[\mathcal{L}]$  is in  $\mathbf{C}$ ; moreover, the query complexity is  $\mathbf{C}$ -hard with respect to a reduction iff some language in  $\text{QINF}[\mathcal{L}]$  is  $\mathbf{C}$ -hard with respect to the reduction, and it is  $\mathbf{C}$ -complete with respect to a reduction iff it is in  $\mathbf{C}$  and it is  $\mathbf{C}$ -hard with respect to the reduction.

**Definition 4.** Denote by  $\text{DINF}[\mathbb{S}, \mathbf{Q}, \mathbf{E}]$  the language consisting of strings  $(N, \gamma)$  for which  $\mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma$  with respect to the grounding of  $\mathbb{S}$  on domain of size  $N$ . Denote by  $\text{DINF}[\mathcal{L}]$  the set of languages  $\text{DINF}[\mathbb{S}, \mathbf{Q}, \mathbf{E}]$  for  $\mathbb{S}$  where the bodies of definition axioms in  $\mathbb{S}$  are formulas in  $\mathcal{L}$ , and where  $\mathbf{Q}$  and  $\mathbf{E}$  are sets of assignments. The domain complexity of  $\mathcal{L}$  is in  $\mathbf{C}$  iff every language in  $\text{DINF}[\mathcal{L}]$  is in  $\mathbf{C}$ ; moreover, the domain complexity is  $\mathbf{C}$ -hard with respect to a reduction iff some language in  $\text{DINF}[\mathcal{L}]$  is  $\mathbf{C}$ -hard with respect to the reduction, and it is  $\mathbf{C}$ -complete with respect to a reduction iff it is in  $\mathbf{C}$  and it is  $\mathbf{C}$ -hard with respect to the reduction.

We conclude this section with a number of observations.

**Combined complexity** The definition of inferential complexity imposes no restriction on the vocabulary; later we will impose bounds on relation arity. We might instead assume that the vocabulary is fixed; in this case we might use the term *combined complexity*, as this is the term employed in finite model theory and database theory to refer to the complexity of model checking when both the formula and the model are given as input, but the vocabulary is fixed [80].

**Lifted inference** We note that query and domain complexities are related respectively to *dqe-liftability* and *domain-liftability*, as defined in the study of lifted inference [66, 65].

The term “lifted inference” is usually attached to algorithms that try to compute inferences involving parvariables without actually producing groundings [68, 90, 105]. A formal definition of lifted inference has been proposed by Van den Broeck [133]: an algorithm is *domain lifted* iff inference runs in polynomial time with respect to  $N$ , for fixed model and query. This definition assumes that  $N$  is given in unary notation; if  $N$  is given in binary notation, the input is of size  $\log N$ , and a domain lifted algorithm may take exponential time. Domain liftability has been extended to *dqe-liftability*, where the inference must run in polynomial time with respect to  $N$  and the query, for fixed model [66].

In short, dqe-liftability means that query complexity is polynomial, while domain-liftability means that domain complexity is polynomial. Deep results have been obtained both on the limits of liftability [66, 65], and on algorithms that attain liftability [9, 135, 67, 94, 124]. We will use several of these results in our later proofs.

We feel that dqe-liftability and domain-liftability are important concepts but they focus only on a binary choice (polynomial versus non-polynomial); our goal here is to map languages and complexities in more detail. As we have mentioned in Section 1, our main goal is to grasp the complexity, however high, of language features.

**Probabilistic databases** Highly relevant material has been produced in the study of probabilistic databases; that is, databases where data may be associated with probabilities. There exist several probabilistic database systems [31, 70, 120, 137, 139]; for instance, the Trio system lets the user indicate that Amy drives an Acura with probability 0.8 [10]. As another example, the NELL system scans text from the web and builds a database of facts, each associated with a number between zero and one [91].

To provide some focus to this overview, we adopt the framework described by Suciu et al. [123]. Consider a set of relations, each implemented as a table. Each tuple in a table may be associated with a probability. These probabilistic tuples are assumed independent (as dependent tuples can be modeled from independent ones [123, Section 2.7.1]). A probabilistic database management system receives a logical formula  $\phi(\vec{x})$  and must determine, using data and probabilities in the tables, the probability  $\mathbb{P}(\phi(\vec{a}))$  for tuples  $\vec{a}$ . The logical formula  $\phi(\vec{x})$  is referred to as the *query*; for example,  $\phi$  may be a *Union of Conjunctive Queries* (a first-order formula with equality, conjunction, disjunction and existential quantification). Note that the word “query” is not used with the meaning usually adopted in the context of Bayesian networks; in probabilistic databases, a query is a formula whose probability is to be computed.

Suppose that all tuples in the table for relation  $X(\vec{x})$  are associated with identical probability value  $\alpha$ . This table can be viewed as the grounding of a parvariable  $X(\vec{x})$  that is associated with the assessment  $\mathbb{P}(X(\vec{x}) = 1) = \alpha$ . Beame et al. say that a probabilistic database is *symmetric* iff each table in the database can be thus associated with a parvariable and a single probabilistic assessment [9].

Now suppose we have a symmetric probabilistic database and a query  $\phi$ . Because the query is itself a logical formula, results on the complexity of computing its probability can be directly mapped to our study of relational Bayesian network specifications. This is pleasant because several deep results have been derived on the complexity probabilistic databases. We later transfer some of those results to obtain the combined and query complexity of specifications based on first-order logic and on fragments of first-order logic with bounded number of logvars. In Section 5 we also comment on *safe queries* and associated dichotomy theorems from the literature on probabilistic databases.

A distinguishing characteristic of research on probabilistic databases is the intricate search for languages that lead to tractable inferences. Some of the main differences between our goals and the goals of research on probabilistic databases are already captured by Suciu et al. when they compare probabilistic databases and probabilistic graphical models [123, Section 1.2.7]: while probabilistic databases deal with simple probabilistic modeling and possibly large volumes of data, probabilistic graphical models encode complex probability models whose purpose is to yield conditional probabilities. As we have already indicated in our previous discussion of liftability, our main goal is to understand the connection between features of a knowledge representation formalism and the resulting complexity.

We are not so focused on finding tractable cases, even though we are obviously looking for them; in fact, later we present tractability results for the  $\text{DLLite}^{\text{nf}}$  language, results that we take to be one of the main contributions of this paper.

**Query or data complexity?** The definition of query complexity (Definition 3) reminds one of *data complexity* as adopted in finite model theory and in database theory [80]. It is thus not surprising that research on probabilistic databases has used the term “data complexity” to mean the complexity when the database is the only input [123].

In the context of Bayesian networks, usually a “query” is a pair  $(\mathbf{Q}, \mathbf{E})$  of assignments. We have adopted such a terminology in this paper. Now if  $\mathbf{Q}$  and  $\mathbf{E}$  contain all available data, there is no real difference between “query” and “data”. Thus we might have adopted the term “data complexity” in this paper as we only discuss queries that contain all available data.<sup>1</sup> However we feel that there are situations where the “query” is not equal to the “data”. For instance, in *probabilistic relational models* one often uses auxiliary grounded relations to indicate which groundings are parents of a given grounding (we return to this in Section 7). And in *probabilistic logic programming* one can use *probabilistic facts* to associate probabilities with specific groundings [44, 103, 118]. In these cases there is a distinction between the “query”  $(\mathbf{Q}, \mathbf{E})$  and the “data” that regulate parts of the grounded Bayesian network.

Consider another possible difference between “query” and “data”. Suppose we have a relational Bayesian network specification, a formula  $\phi$  whose probability  $\mathbb{P}(\phi)$  is to be computed, and a table with the probabilities for various groundings. Here  $\phi$  is the “query” and the table is the “data” (this sort of arrangement has been used in description logics [20]). One might then either fix the specification and vary the query and the data (“query” complexity), or fix the specification and the query and vary the data (“data” complexity).

It is possible that such distinctions between “query” and “data” are not found to be of practical value in future work. For now we prefer to keep open the possibility of a fine-grained analysis of complexity, so we use the term “query complexity” even though our queries are simply sets of assignments containing all available data.

## 5 The complexity of relational Bayesian network specifications

We start with *function-free first-order logic with equality*, a language we denote by  $\text{FFFO}$ . One might guess that such a powerful language leads to exponentially hard inference problems. Indeed:

**Theorem 4.**  *$\text{INF}[\text{FFFO}]$  is PEXP-complete with respect to many-one reductions, regardless of whether the domain is specified in unary or binary notation.*

We note that Grove, Halpern and Koller have already argued that counting the number of suitably defined distinct interpretations of monadic first-order logic is hard for the class of languages decided by exponential-time counting Turing machines [57, Theorem 4.14]. As they do not present a proof of their counting result (and no similar proof seems to be available in the literature), and as we need some of the reasoning to address query complexity later, we present a detailed proof of Theorem 4 in A.

We emphasize that when the domain is specified in binary notation the proof of Theorem 4 only requires relations of arity one. One might hope to find lower complexity classes for fragments of  $\text{FFFO}$  that go beyond monadic logic but restrict quantification. For instance, the popular description logic  $\mathcal{ALC}$  restricts quantification to obtain PSPACE-completeness of satisfiability [3]. Inspired by this result, we might consider the following specification language:

**Definition 5.** *The language  $\text{ALC}$  consists of all formulas recursively defined so that  $X(\chi)$  is a formula where  $X$  is a unary relation,  $\neg\phi$  is a formula when  $\phi$  is a formula,  $\phi \wedge \varphi$  is a formula when both  $\phi$  and*

---

<sup>1</sup>In fact we have used the term *data complexity* in previous work [26].

$\varphi$  are formulas, and  $\exists y : X(x, y) \wedge Y(y)$  is a formula when  $X$  is a binary relation and  $Y$  is a unary relation.

However, ALC does not move us below PEXP when domain size is given in binary notation:

**Theorem 5.** *INF[ALC] is PEXP-complete with respect to many-one reductions, when domain size is given in binary notation.*

Now returning to full FFFO, consider its query complexity. We divide the analysis in two parts, as the related proofs are quite different:<sup>2</sup>

**Theorem 6.** *QINF[FFFO] is PEXP-complete with respect to many-one reductions, when the domain is specified in binary notation.*

**Theorem 7.** *QINF[FFFO] is PP-complete with respect to many-one reductions when the domain is specified in unary notation.*

As far as domain complexity is concerned, it seems very hard to establish a completeness result for FFFO when domain size is given in binary notation.<sup>3</sup> We simply rephrase an ingenious argument by Jaeger [65] to establish:

**Theorem 8.** *Suppose  $\text{NETIME} \neq \text{ETIME}$ . Then DINF[FFFO] is not solved in deterministic exponential time, when the domain size is given in binary notation.*

And for domain size in unary notation:

**Theorem 9.** *DINF[FFFO] is  $\text{PP}_1$ -complete with respect to many-one reductions, when the domain is given in unary notation.*

Theorem 9 is in essence implied by a major result by Beame et al. [9, Lemma 3.9]: they show that counting the number of interpretations for formulas in the *three-variable* fragment FFFO<sup>3</sup> is  $\#P_1$ -complete. The fragment FFFO<sup>k</sup> consists of the formulas in FFFO that employ at most  $k$  logvars (note that logvar symbols may be reused within a formula, but there is a bounded supply of such symbols) [80, Chapter 111]. The proof by Beame et al. is rather involved because they are restricted to three logvars; in A we show that a relatively simple proof of Theorem 9 is possible when there is no bound on the number of logvars, a small contribution that may be useful to researchers.

It is apparent from Theorems 4, 5, 6 and 8 that we are bound to obtain exponential complexity when domain size is given in binary notation. Hence, from now on we work with domain sizes in unary notation, unless explicitly indicated.

Of course, a domain size in unary notation cannot by itself avoid exponential behavior, as an exponentially large number of groundings can be simulated by increasing arity. For instance, a domain with two individuals leads to  $2^k$  groundings for a relation with arity  $k$ . Hence, we often assume that our relations have bounded arity. We might instead assume that the vocabulary is fixed, as done in finite model theory when studying combined complexity. We prefer the more general strategy where we bound arity; clearly a fixed vocabulary implies a fixed maximum arity.

With such additional assumptions, we obtain PSPACE-completeness of inferential complexity. With a few differences, this result is implied by results by Beame et al. in their important paper [9, Theorem 4.1]: they show that counting interpretations with a fixed vocabulary is PSPACE-complete (that is, they focus on combined complexity and avoid conditioning assignments). We present a short proof of Theorem 10 within our framework in A.

<sup>2</sup>The query complexity of *monadic* FFFO seems to be open, both for domain in binary and in unary notation; proofs of Theorems 6 and 7 need relations of arity two.

<sup>3</sup>One might think that, when domain size is given in binary notation, some small change in the proof of Theorem 9 would show that DINF[FFFO] is complete for a suitable subset of PEXP. Alas, it does not seem easy to define a complexity class that can convey the complexity of DINF[FFFO] when domain size is in binary notation. Finding the precise complexity class of DINF[FFFO] is an open problem.

**Theorem 10.**  $\text{INF}[\text{FFFO}]$  is PSPACE-complete with respect to many-one reductions, when relations have bounded arity and the domain size is given in unary notation.

Note that the proof of Theorem 7 is already restricted to arity 2, hence  $\text{QINF}[\text{FFFO}]$  is PP-complete with respect to many-one reductions when relations have bounded arity (larger than one) and the domain is given in unary notation.

We now turn to  $\text{FFFO}^k$ . As we have already noted, this sort of language has been studied already, again by Beame et al., who have derived their domain and combined complexity [9]. In A we present a short proof of the next result, to emphasize that it follows by a simple adaptation of the proof of Theorem 7:

**Theorem 11.**  $\text{INF}[\text{FFFO}^k]$  is PP-complete with respect to many-one reductions, for all  $k \geq 0$ , when the domain size is given in unary notation.

Query complexity also follows directly from arguments in the proofs of previous results, as is clear from the proof of the next theorem in A:<sup>4</sup>

**Theorem 12.**  $\text{QINF}[\text{FFFO}^k]$  is PP-complete with respect to many-one reductions, for all  $k \geq 2$ , when domain size is given in unary notation.

Now consider domain complexity for the bounded variable fragment; previous results in the literature establish this complexity [9, 133, 135]. In fact, the case  $k > 2$  is based on a result by Beame et al. that we have already alluded to; in A we present a simplified argument for this result.

**Theorem 13.**  $\text{DINF}[\text{FFFO}^k]$  is  $\text{PP}_1$ -complete with respect to many-one reductions, for  $k > 2$ , and is in P for  $k \leq 2$ , when the domain size is given in unary notation.

There are important knowledge representation formalisms within bounded-variable fragments of FFFO. An example is the description logic  $\mathcal{ALC}$  that we have discussed before: every sentence in this description logic can be translated to a formula in  $\text{FFFO}^2$  [3]. Hence we obtain:

**Theorem 14.** Suppose the domain size is specified in unary notation. Then  $\text{INF}[\text{ALC}]$  and  $\text{QINF}[\text{ALC}]$  are PP-complete, and  $\text{DINF}[\text{ALC}]$  is in P.

As a different exercise, we now consider the quantifier-free fragment of FFFO. In such a language, every logvar in the body of a definition axiom must appear in the defined relation, as no logvar is bound to any quantifier. Denote this language by QF; in Section 7 we show the close connection between QF and *plate models*. We have:

**Theorem 15.** Suppose relations have bounded arity.  $\text{INF}[\text{QF}]$  and  $\text{QINF}[\text{QF}]$  are PP-complete with respect to many-one reductions, and  $\text{DINF}[\text{QF}]$  requires constant computational effort. These results hold even if domain size is given in binary notation.

As we have discussed at the end of Section 4, the literature on lifted inference and on probabilistic databases has produced deep results on query and domain complexity. One example is the definition of *safe queries*, a large class of formulas with tractable query complexity [32]. Similar classes of formulas have been studied for symmetric probabilistic databases [56]. Based on such results in the literature, one might define the language SAFE consisting of safe queries, or look for similar languages with favorable query complexity. We prefer to move to description logics in the next section, leaving safe queries and related languages to future work; we prefer to focus on languages whose complexity can be determined directly from their constructs (note that a sentence can be decided to be safe in polynomial time, but such a decision requires computational support).

---

<sup>4</sup>The case  $k = 1$  seems to be open; when  $k = 1$ , query complexity is polynomial when inference is solely on unary relations [134, 135]. When  $k = 0$  we obtain propositional networks and then query complexity is polynomial by Theorem 3.

## 6 Specifications based on description logics: the DDLite language

The term “description logic” encompasses a rich family of formal languages with the ability to encode terminologies and assertions about individuals. Those languages are now fundamental knowledge representation tools, as they have solid semantics and computational guarantees concerning reasoning tasks [3]. Given the favorable properties of description logics, much effort has been spent in mixing them with probabilities [83].

In this section we examine relational Bayesian network specifications based on description logics. Such specifications can benefit from well tested tools and offer a natural path to encode probabilistic ontologies. Recall that we have already examined the description logic  $\mathcal{ALC}$  in the previous section.

Typically a description logic deals with *individuals*, *concepts*, and *roles*. An individual like *John* corresponds to a constant in first-order logic; a concept like *researcher* corresponds to a unary relation in first-order logic; and a role like *buysFrom* corresponds to a binary relation in first-order logic. A vocabulary contains a set of individuals plus some *primitive* concepts and some *primitive* roles. From these primitive concepts and roles one can define other concepts and roles using a set of operators. For instance, one may allow for concept *intersection*: then  $C \sqcap D$  is the intersection of concepts  $C$  and  $D$ . Likewise,  $C \sqcup D$  is the *union* of  $C$  and  $D$ , and  $\neg C$  is the *complement* of  $C$ . For a role  $r$  and a concept  $C$ , a common construct is  $\forall r.C$ , called a *value restriction*. Another common construct is  $\exists r.C$ , an *existential restriction*. Description logics often define composition of roles, inverses of roles, and even intersection/union/complement of roles. For instance, usually  $r^-$  denotes the *inverse* of role  $r$ .

The semantics of description logics typically resorts to domains and interpretations. A *domain*  $\mathcal{D}$  is a set. An *interpretation*  $\mathbb{I}$  maps each individual to an element of the domain, each primitive concept to a subset of the domain, and each role to a set of pairs of elements of the domain. And then the semantics of  $C \sqcap D$  is fixed by  $\mathbb{I}(C \sqcap D) = \mathbb{I}(C) \cap \mathbb{I}(D)$ . Similarly,  $\mathbb{I}(C \sqcup D) = \mathbb{I}(C) \cup \mathbb{I}(D)$  and  $\mathbb{I}(\neg C) = \mathcal{D} \setminus \mathbb{I}(C)$ . And for the restricted quantifiers, we have  $\mathbb{I}(\forall r.C) = \{x \in \mathcal{D} : \forall y : (x, y) \in \mathbb{I}(r) \rightarrow y \in \mathbb{I}(C)\}$  and  $\mathbb{I}(\exists r.C) = \{x \in \mathcal{D} : \exists y : (x, y) \in \mathbb{I}(r) \wedge y \in \mathbb{I}(C)\}$ . The semantics of the inverse role  $r^-$  is, unsurprisingly, given by  $\mathbb{I}(r^-) = \{(x, y) \in \mathcal{D} \times \mathcal{D} : (y, x) \in \mathbb{I}(r)\}$ .

We can translate this syntax and semantics to their counterparts in first-order logic. Thus  $C \sqcap D$  can be read as  $C(x) \wedge D(x)$ ,  $C \sqcup D$  as  $C(x) \vee D(x)$ , and  $\neg C$  as  $\neg C(x)$ . Moreover,  $\forall r.C$  translates into  $\forall y : r(x, y) \rightarrow C(y)$  and  $\exists r.C$  translates into  $\exists y : r(x, y) \wedge C(y)$ .

Definition 5 introduced the language  $\mathcal{ALC}$  by adopting intersection, complement, and existential restriction (union and value restrictions are then obtained from the other constructs). We can go much further than  $\mathcal{ALC}$  in expressivity and still be within the two-variable fragment of FFFO; for instance, we can allow for role composition, role inverses, and Boolean operations on roles. The complexity of such languages is obtained from results discussed in the previous section.

Clearly we can also contemplate description logics that are less expressive than  $\mathcal{ALC}$  in an attempt to obtain tractability. Indeed, some description logics combine selected Boolean operators with restricted quantification to obtain polynomial complexity of logical inferences. Two notable such description logics are  $\mathcal{EL}$  and DL-Lite; due to their favorable balance between expressivity and complexity, they are the basis of existing standards for knowledge representation.<sup>5</sup>

Consider first the description logic  $\mathcal{EL}$ , where the only allowed operators are intersection and existential restrictions, and where the *top* concept is available, interpreted as the whole domain [2]. Note that we can translate every sentence of  $\mathcal{EL}$  into the negation-free fragment of  $\mathcal{ALC}$ , and we can simulate the *top* concept with the assessment  $\mathbb{P}(\top = 1) = 1$ . Thus we take the language  $\mathcal{EL}$  as the negation-free fragment of  $\mathcal{ALC}$ . Because  $\mathcal{EL}$  contains conjunction, we easily have that  $\text{INF}[\mathcal{EL}]$  is PP-hard by Theorem 2. And domain complexity is polynomial as implied by  $\text{DINF}[\mathcal{ALC}]$ . Query complexity requires some additional work as discussed in A; altogether, we have:<sup>6</sup>

<sup>5</sup>Both  $\mathcal{EL}$  and DL-Lite define standard profiles of the OWL knowledge representation language, as explained at <http://www.w3.org/TR/owl2-profiles/>.

<sup>6</sup>The proof of Theorem 16 uses queries with negative assignments; both the inferential/query complexity of  $\mathcal{EL}$  are open when the query is restricted to positive assignments.



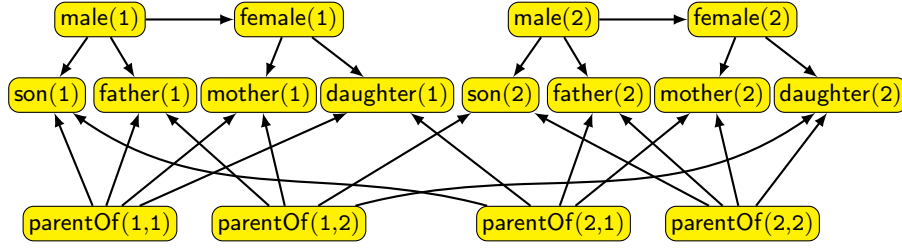


Figure 6: Grounding of relational Bayesian network specification from Example 6 on domain  $\mathcal{D} = \{1, 2\}$ .

**Theorem 16.** *Suppose the domain size is specified in unary notation. Then  $\text{INF}[\text{EL}]$  and  $\text{QINF}[\text{EL}]$  are PP-complete with respect to many-one reductions, even if the query contains only positive assignments, and  $\text{DINF}[\text{EL}]$  is in P.*

We can present more substantial results when we focus on the negation-free fragment of the popular description logic DL-Lite [19]. DL-Lite is particularly interesting because it captures central features of ER or UML diagrams, and yet common inference services have polynomial complexity [1].

The simplicity and computational efficiency of the DL-Lite language have led many researchers to mix them with probabilities. For instance, D’Amato et al. [33] propose a variant of DL-Lite where the interpretation of each sentence is conditional on a context that is specified by a Bayesian network. A similar approach was taken by Ceylan and Peñalosa [22], with minor semantic differences. A different approach is to extend the syntax of DL-Lite sentences with probabilistic subsumption connectives, as in the *Probabilistic DL-Lite* [113]. Differently from our focus here, none of those proposals employ DL-Lite to specify Bayesian networks.

In DL-Lite one has primitive concepts as before, and also *basic concepts*: a basic concept is either a primitive concept, or  $\exists r$  for a role  $r$ , or  $\exists r^-$  for a role  $r$ . Again,  $r^-$  denotes the inverse of  $r$ . And then a concept in DL-Lite is either a basic concept, or  $\neg C$  when  $C$  is a basic concept, or  $C \sqcap D$  when  $C$  and  $D$  are concepts. The semantics of  $r^-$ ,  $\neg C$  and  $C \sqcap D$  are as before, and the semantics of  $\exists r$  is, unsurprisingly, given by  $\mathbb{I}(\exists r) = \{x \in \mathcal{D} : \exists y : (x, y) \in \mathbb{I}(r)\}$ .

We focus on the negation-free fragment of DL-Lite; that is, we consider:

**Definition 6.** *The language  $\text{DLLite}^{\text{nf}}$  consists of all formulas recursively defined so that  $X(\chi)$  are formulas when  $X$  is a unary relation,  $\phi \wedge \varphi$  is a formula when both  $\phi$  and  $\varphi$  are formulas, and  $\exists y : X(\chi, y)$  and  $\exists y : X(y, \chi)$  are formulas when  $X$  is a binary relation.*

**Example 6.** The following definition axioms express a few facts about families:

$$\begin{aligned} \text{female}(\chi) &\equiv \neg \text{male}(\chi), \\ \text{father}(\chi) &\equiv \text{male}(\chi) \wedge \exists y : \text{parentOf}(\chi, y), \\ \text{mother}(\chi) &\equiv \text{female}(\chi) \wedge \exists y : \text{parentOf}(\chi, y), \\ \text{son}(\chi) &\equiv \text{male}(\chi) \wedge \exists y : \text{parentOf}(y, \chi), \\ \text{daughter}(\chi) &\equiv \text{female}(\chi) \wedge \exists y : \text{parentOf}(y, \chi). \end{aligned}$$

For domain  $\mathcal{D} = \{1, 2\}$ , this relational Bayesian network is grounded into the Bayesian network in Figure 6.  $\square$

We again have that  $\text{INF}[\text{DLLite}^{\text{nf}}]$  is PP-hard by Theorem 2. However, inferential complexity becomes polynomial when the query is positive:

**Theorem 17.** *Suppose the domain size is specified in unary notation. Then  $\text{DINF}[\text{DLLite}^{\text{nf}}]$  is in P; also,  $\text{INF}[\text{DLLite}^{\text{nf}}]$  and  $\text{QINF}[\text{DLLite}^{\text{nf}}]$  are in P when the query  $(\mathbf{Q}, \mathbf{E})$  contains only positive assignments.*

Language ( $N$ in unary notation)	Inferential	Query	Domain
$\text{Prop}(\wedge)$ , positive query	P	P	—
$\text{Prop}(\wedge, \neg)$ , $\text{Prop}(\wedge)$ , $\text{Prop}(\vee)$	PP	P	—
FFFO	PEXP	PP	$\text{PP}_1$
FFFO with bound on relation arity	PSPACE	PP	$\text{PP}_1$
$\text{FFFO}^k$ with $k \geq 3$	PP	PP	$\text{PP}_1$
QF with bound on arity	PP	PP	P
ALC	PP	PP	P
EL	PP	PP	P
$\text{DLLite}^{\text{nf}}$	PP	PP	P
$\text{DLLite}^{\text{nf}}$ , positive query	P	P	P

Table 1: Inferential, query and domain complexity for relational Bayesian networks based on various logical languages with domain size given in unary notation. All cells indicate completeness with respect to many-one reductions. On top of these results, note that when domain size is given in binary notation we have, with respect to many-one reductions:  $\text{INF}[\text{FFFO}]$  is PEXP-complete (even when restricted to relations of arity 1),  $\text{QINF}[\text{FFFO}]$  is PEXP-complete (even when restricted to relations of arity 2), and  $\text{INF}[\text{ALC}]$  is PEXP-complete.

In proving this result (in A) we show that an inference with a positive query can be reduced to a particular tractable model counting problem. The analysis of this model counting problem is a result of independent interest.

Using the model counting techniques we just mentioned, we can also show that a related problem, namely finding the most probable explanation, is polynomial for relational Bayesian network specifications based on  $\text{DLLite}^{\text{nf}}$ . To understand this, consider a relational Bayesian network  $\mathbb{S}$  based on  $\text{DLLite}^{\text{nf}}$ , a set of assignments  $\mathbf{E}$  for ground atoms, and a domain size  $N$ . Denote by  $\mathbf{X}$  the set of random variables that correspond to groundings of relations in  $\mathbb{S}$ . Now there is at least one set of assignments  $\mathbf{M}$  such that: (i)  $\mathbf{M}$  contains assignments to all random variables in  $\mathbf{X}$  that are not mentioned in  $\mathbf{E}$ ; and (ii)  $\mathbb{P}(\mathbf{M}, \mathbf{E})$  is maximum over all such sets of assignments. Denote by  $\text{MLE}(\mathbb{S}, \mathbf{E}, N)$  the function problem that consists in generating such a set of assignments  $\mathbf{M}$ .

**Theorem 18.** *Given a relational Bayesian network  $\mathbb{S}$  based on  $\text{DLLite}^{\text{nf}}$ , a set of positive assignments to grounded relations  $\mathbf{E}$ , and a domain size  $N$  in unary notation,  $\text{MLE}(\mathbb{S}, \mathbf{E}, N)$  can be solved in polynomial time.*

These results on  $\text{DLLite}^{\text{nf}}$  can be directly extended in some other important ways. For example, suppose we allow negative groundings of roles in the query. Then most of the proof of Theorem 17 follows (the difference is that the intersection graphs used in the proof do not satisfy the same symmetries); we can then resort to approximations for weighted edge cover counting [82], so as to develop a fully polynomial-time approximation scheme (FPTAS) for inference. Moreover, the  $\text{MLE}(\mathbb{S}, \mathbf{E}, N)$  problem remains polynomial. Similarly, we could allow for different groundings of the same relation to be associated with different probabilities; the proofs given in A can be modified to develop a FPTAS for inference.

We have so far presented results for a number of languages. Table 1 summarizes most of our findings; as noted previously, most of the results on FFFO with bound on relation arity and on  $\text{FFFO}^k$  have been in essence derived by Beame et al. [9].

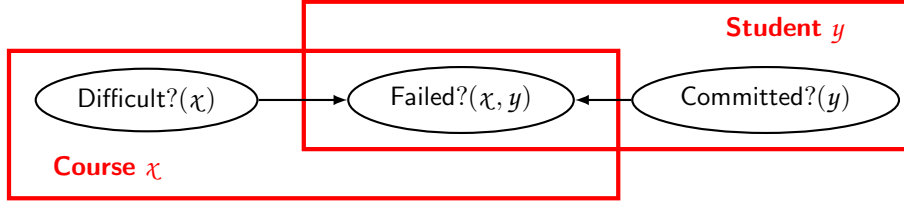


Figure 7: Plate model for the University World. We show logvars explicitly, even though they are not always depicted in plate models.

## 7 Plates, probabilistic relational models, and related specification languages

In this paper we have followed a strategy that has long been cherished in the study of formal languages; that is, we have focused on languages that are based on minimal sets of constructs borrowed from logic. Clearly this plan succeeds only to the extent that results can be transferred to practical specification languages. In this section we examine some important cases where our strategy pays off.

Consider, for instance, *plate models*, a rather popular specification formalism. Plate models have been extensively used in statistical practice [84] since they were introduced in the BUGS project [50, 85]. In machine learning, they have been used to convey several models since their first appearance [17].

There seems to be no standard formalization for plate models, so we adapt some of our previous concepts as needed. A plate model consists of a set of parvariables, a directed acyclic graph where each node is a parvariable, and a set of *template conditional probability distributions*. Parvariables are *typed*: each parameter of a parvariable is associated with a set, the *domain* of the parameter. All parvariables that share a domain are said to belong to a *plate*. The central constraint on “standard” plate models is that the domains that appear in the parents of a parvariable must appear in the parvariable. For a given parvariable  $X$ , its corresponding template conditional probability distribution associates a probability value to each value of  $X$  given each configuration of parents of  $X$ .

To make things simple, here we focus on parvariables that correspond to relations, thus every random variable has values **true** and **false** (plate models in the literature often specify discrete and even continuous random variables [84, 122]). Our next complexity results do not really change if one allows parvariables to have a finite number of values.

We can use the same semantics as before to interpret plate models, with a small change: now the groundings of a relation are produced by running only over the domains of its associated logvars.

**Example 7.** Suppose we are interested in a “University World” containing a population of students and a population of courses [47]. Parvariable  $\text{Failed?}(\chi, y)$  yields the final status of student  $y$  in course  $\chi$ ;  $\text{Difficult?}(\chi)$  is a parvariable indicating the difficulty of a course  $\chi$ , and  $\text{Committed?}(y)$  is a parvariable indicating the commitment of student  $y$ .

A plate model is drawn in Figure 7, where plates are rectangles. Each parvariable is associated with a template conditional probability distribution:

$$\mathbb{P}(\text{Difficult?}(\chi) = 1) = 0.3, \quad \mathbb{P}(\text{Committed?}(y) = 1) = 0.7,$$

$$\mathbb{P}\left(\text{Failed?}(\chi, y) = 1 \mid \begin{array}{l} \text{Difficult?}(\chi) = d, \\ \text{Committed?}(y) = c \end{array}\right) = \begin{cases} 0.4 & \text{if } d = 0, c = 0; \\ 0.2 & \text{if } d = 0, c = 1; \\ 0.9 & \text{if } d = 1, c = 0; \\ 0.8 & \text{if } d = 1, c = 1. \end{cases} \quad \square$$

Note that plate models can always be specified using definition axioms in the quantifier-free fragment of FFFO, given that the logvars of a relation appear in its parent relations. For instance, the

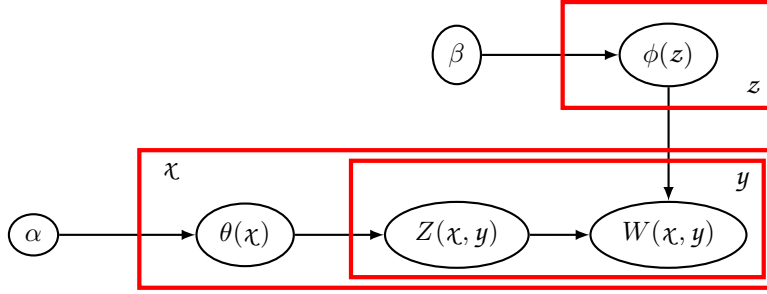


Figure 8: Smoothed Latent Dirichlet Allocation.

table in Example 7 can be encoded as follows:

$$\text{Failed?}(\chi, y) \equiv \left( \begin{array}{l} (\neg \text{Difficult?}(\chi) \wedge \neg \text{Committed?}(y) \wedge A_1(\chi, y)) \vee \\ (\neg \text{Difficult?}(\chi) \wedge \text{Committed?}(y) \wedge A_2(\chi, y)) \vee \\ (\text{Difficult?}(\chi) \wedge \neg \text{Committed?}(y) \wedge A_3(\chi, y)) \vee \\ (\text{Difficult?}(\chi) \wedge \text{Committed?}(y) \wedge A_4(\chi, y)) \end{array} \right), \quad (5)$$

where we introduced four auxiliary parvariables with associated assessments  $\mathbb{P}(A_1(\chi, y) = 1) = 0.4$ ,  $\mathbb{P}(A_2(\chi, y) = 1) = 0.2$ ,  $\mathbb{P}(A_3(\chi, y) = 1) = 0.9$ , and  $\mathbb{P}(A_4(\chi, y) = 1) = 0.8$ .

Denote by  $\text{INF}[\text{PLATE}]$  the language consisting of inference problems as in Definition 2, where relational Bayesian network specifications are restricted to satisfy the constraints of plate models. Adopt  $\text{QINF}[\text{PLATE}]$  and  $\text{DINF}[\text{PLATE}]$  similarly. We can reuse arguments in the proof of Theorem 15 to show that:

**Theorem 19.**  *$\text{INF}[\text{PLATE}]$  and  $\text{QINF}[\text{PLATE}]$  are PP-complete with respect to many-one reductions, and  $\text{DINF}[\text{PLATE}]$  requires constant computational effort. These results hold even if the domain size is given in binary notation.*

One can find extended versions of plate models in the literature, where a node can have children in other plates (for instance the *smoothed Latent Dirichlet Allocation* (sLDA) model [12] depicted in Figure 8). In such extended plates a template conditional probability distribution can refer to logvars from plates that are not enclosing the parvariable; if definition axioms are then allowed to specify template distributions, one obtains as before  $\text{INF}[\text{FFFO}]$ ,  $\text{QINF}[\text{FFFO}]$ , etc; that is, results obtained in previous sections apply.

Besides plates, several other languages can encode repetitive Bayesian networks. Early proposals resorted to object orientation [73, 86], to frames [74], and to rule-based statements [5, 52, 58], all inspired by knowledge-based model construction [61, 53, 138]. Some of these proposals coalesced into a family of models loosely grouped under the name of *Probabilistic Relational Models* (PRMs) [46]. We adopt PRMs as defined by Getoor et al. [47]; again, to simplify matters, we focus on parvariables that correspond to relations.

Similarly to a plate model, a PRM contains typed parvariables and domains. A domain is now called a *class*; each class appears as a box containing parvariables. For instance, Figure 9 depicts a PRM for the University World: edges between parvariables indicate probabilistic dependence, and dashed edges between classes indicate *associations* between elements of the classes. In Figure 9 we have classes **Course**, **Student**, and **Registration**, with associations between them. Consider association **studentOf**: the idea is that **studentOf**( $\chi, z$ ) holds when  $\chi$  is the student in registration  $z$ . Following terminology by Koller and Friedman [71], we say that relations that encode classes and associations, such as **Course** and **studentOf**, are *guard parvariables*.

A *relational skeleton* for a PRM is an explicit specification of elements in each class, plus the explicit specification of pairs of objects that are associated. That is, the relational skeleton specifies the groundings of the guard parvariables.

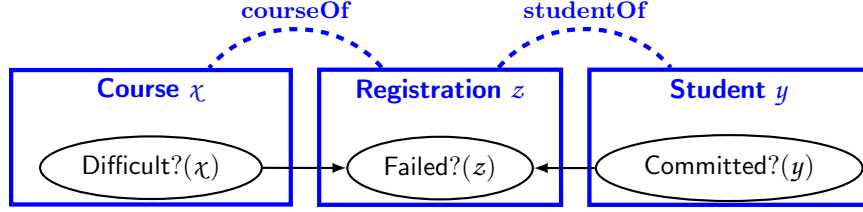


Figure 9: PRM for the University World. We show logvars explicitly, even though they are not always depicted in PRMs. Associations appear as dashed edges [47, 121].

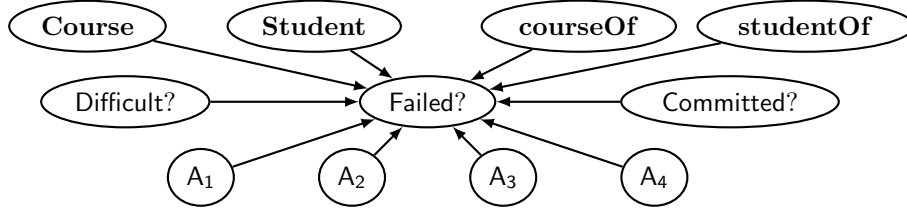


Figure 10: Parvariable graph for the University World PRM.

Each parvariable  $X$  in a PRM is then associated with a *template probability distribution* that specifies probabilities for the parvariable  $X$  given a selected set of other parvariables. The latter are the *parents* of  $X$ , again denoted by  $\text{pa}(X)$ . In the University World of Figure 9, we must associate with **Failed?** the template probabilities for  $\mathbb{P}(\text{Failed?}(z) | \text{Difficult?}(\chi), \text{Committed?}(y))$ . But differently from plate models, here the parents of a particular grounding are determined by going through associations: for instance, to find the parents of **Failed?}(r)**, we must find the course  $c$  and the student  $s$  such that **courseOf}(c, r)** and **studentOf}(s, r)** hold, and then we have parents **Difficult?}(c)** and **Committed?}(s)**.

All of these types and associations can, of course, be encoded using first-order logic, as long as all parvariables correspond to relations. For instance, here is a definition axiom that captures the PRM for the University World:

$$\text{Failed?}(z) \equiv \forall \chi : \forall y : \left( \begin{array}{l} \text{Course}(\chi) \wedge \text{Student}(y) \wedge \\ \text{courseOf}(\chi, z) \wedge \text{studentOf}(y, z) \end{array} \right) \rightarrow \left( \begin{array}{l} (\neg \text{Difficult?}(\chi) \wedge \neg \text{Committed?}(y) \wedge A_1(\chi, y)) \vee \\ (\neg \text{Difficult?}(\chi) \wedge \text{Committed?}(y) \wedge A_2(\chi, y)) \vee \\ (\text{Difficult?}(\chi) \wedge \neg \text{Committed?}(y) \wedge A_3(\chi, y)) \vee \\ (\text{Difficult?}(\chi) \wedge \text{Committed?}(y) \wedge A_4(\chi, y)) \end{array} \right),$$

using the same auxiliary parvariables employed in Expression (5). The parvariable graph for the resulting specification is depicted in Figure 10.

Thus we can take a PRM and translate it into a relational Bayesian network specification  $\mathbb{S}$ . As long as the parvariable graph is acyclic, results in the previous sections apply. To see this, note that a skeleton is simply an assignment for all groundings of the guard parvariables. Thus a skeleton can be encoded into a set of assignments  $\mathbf{S}$ , and our inferences should focus on deciding  $\mathbb{P}(\mathbf{Q} | \mathbf{E}, \mathbf{S}) > \gamma$  with respect to  $\mathbb{S}$  and a domain that is the union of all classes of the PRM.

For instance, suppose we have a fixed PRM and we receive as input a skeleton and a query  $(\mathbf{Q}, \mathbf{E})$ , and we wish to decide whether  $\mathbb{P}(\mathbf{Q} | \mathbf{E}) > \gamma$ . If the template probability distributions are specified with FFFO, and the parvariable graph is acyclic, then this decision problem is a PP-complete problem. We can replay our previous results on inferential and query complexity this way. The concept of domain complexity seems less meaningful when PRMs are considered: the larger the domain, the more data on guard parvariables are needed, so we cannot really fix the domain in isolation.

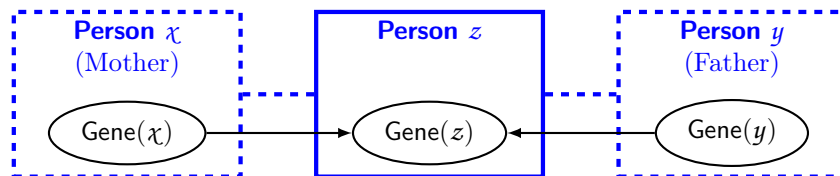


Figure 11: A PRM for the blood-type model, adapted from a proposal by Getoor et al. [47]. Dashed boxes stand for repeated classes; Getoor et al. suggest that some associations may be constrained to be “guaranteed acyclic” so that the whole model is consistent for any skeleton that satisfies the constraints.

We conclude this section with a few observations.

**Cyclic parvariable graphs** Our results assume acyclicity of parvariable graphs, but this is not a condition that is typically imposed on PRMs. A cyclic parvariable graph may still produce an acyclic grounding, depending on the given skeleton. For instance, one might want to model blood-type inheritance, where a **Person** inherits a genetic predisposition from another **Person**. This creates a loop around the class **Person**, even though we do not expect a cycle in any valid grounding of the PRM. The literature has proposed languages that allow cycles [47, 60]; one example is shown in Figure 11. The challenge then is to guarantee that a given skeleton will lead to an acyclic grounded Bayesian network; future work on cyclic parvariable graphs must deal with such a consistency problem.

**Other specification languages** There are several other languages that specify PRMs and related formalisms; such languages can be subjected to the same analysis we have explored in this paper. A notable formalism is the Probabilistic Relational Language (PRL) [48], where logic programs are used to specify PRMs; the specification is divided into logical background (that is, guard parvariables), probabilistic background, and probabilistic dependencies. Two other examples of textual formalisms that can be used to encode PRMs are Logical Bayesian Networks (LBNs) [43, 42] and Bayesian Logic Programs (BLPs) [69, 112]. Both distinguish between *logical* predicates that constrain groundings (that is, guard parvariables), and *probabilistic* or *Bayesian* predicates that encode probabilistic assessments [93].

A more visual language, based on Entity-Relationship Diagrams, is DAPER [60]. Figure 12 shows a DAPER diagram for the University World and a DAPER diagram for the blood-type model. Another diagrammatic language is given by Multi-Entity Bayesian Networks (MEBNs), a graphical representation for arbitrary first-order sentences [78]. Several other graphical languages mix probabilities with description logics [21, 24, 39, 72], as we have mentioned in Section 6.

There are other formalisms in the literature that are somewhat removed from our framework. For instance, Jaeger’s *Relational Bayesian Networks* [62, 63] offer a solid representation language where the user can directly specify and manipulate probability values, for instance specifying that a probability value is the average of other probability values. We have examined the complexity of Relational Bayesian Networks elsewhere [88]; some results and proofs, but not all of them, are similar to the ones presented here. There are also languages that encode repetitive Bayesian networks using functional programming [87, 89, 125, 102] or logic programming [25, 44, 103, 104, 114, 117]. We have examined the complexity of the latter formalisms elsewhere [27, 28, 29]; again, some results and proofs, but not all of them, are similar to the ones presented here.

## 8 A detour into Valiant’s counting hierarchy

We have so far focused on inferences that compare a conditional probability with a given rational number. However one might argue that the real purpose of a probabilistic inference is to compute a

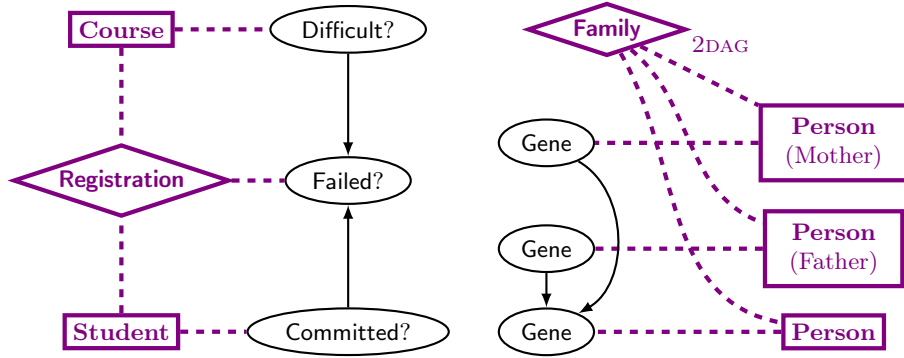


Figure 12: Left: A DAPER diagram for the University World. Right: A DAPER diagram for the blood-type model, as proposed by Heckerman et al. [60]; note the constraint 2DAG, meaning that each child of the node has at most two parents and cannot be his or her own ancestor.

probability value. We can look at the complexity of calculating such numbers using Valiant’s counting classes and their extensions. Indeed, most work on probabilistic databases and lifted inference has used Valiant’s classes. In this section we justify our focus on decision problems, and adapt our results to Valiant’s approach.

Valiant defines, for complexity class  $A$ , the class  $\#A$  to be  $\cup_{\mathcal{L} \in A} (\#P)^{\mathcal{L}}$ , where  $(\#P)^{\mathcal{L}}$  is the class of functions counting the accepting paths of nondeterministic polynomial time Turing machines with  $\mathcal{L}$  as oracle [130]. Valiant declares function  $f$  to be  $\#P$ -hard when  $\#P \subseteq FP^f$ ; that is,  $f$  is  $\#P$ -hard if any function  $f'$  in  $\#P$  can be reduced to  $f$  by the analogue of a polynomial time Turing reduction (recall that  $FP$  is the class of functions that can be computed in polynomial time by a deterministic Turing machine). Valiant’s is a very loose notion of hardness; as shown by Toda and Watanabe [128], any function in  $\#PH$  can be reduced to a function in  $\#P$  via a one-Turing reduction (where  $\#PH$  is a counting class with the whole polynomial hierarchy as oracle). Thus a Turing reduction is too weak to distinguish classes of functions within  $\#PH$ . For this reason, other reductions have been considered for counting problems [8, 41].

A somewhat stringent strategy is to say that  $f$  is  $\#P$ -hard if any function  $f'$  in  $\#P$  can be produced from  $f$  by a *parsimonious* reduction; that is,  $f'(\ell)$  is computed by applying a polynomial time function  $g$  to  $x$  and then computing  $f(g(\ell))$  [119]. However, such a strategy is inadequate for our purposes: counting classes such as  $\#P$  produce integers, and we cannot produce integers by computing probabilities.

A sensible strategy is to adopt a reduction that allows for multiplication by a polynomial function. This has been done both in the context of probabilistic inference with “reductions modulo normalization” [75] and in the context of probabilistic databases [123]. We adopt the reductions proposed by Bulatov et al. in their study of weighted constraint satisfaction problems [16]. They define *weighted reductions* as follows (Bulatov et al. consider functions into the algebraic numbers, but for our purposes we can restrict the weighted reductions to rational numbers):

**Definition 7.** Consider functions  $f_1$  and  $f_2$  from an input language  $\mathcal{L}$  to rational numbers  $\mathbb{Q}$ . A weighted reduction from  $f_1$  to  $f_2$  is a pair of polynomial time functions  $g_1 : \mathcal{L} \rightarrow \mathbb{Q}$  and  $g_2 : \mathcal{L} \rightarrow \mathcal{L}$  such that  $f_1(\ell) = g_1(\ell)f_2(g_2(\ell))$  for all  $\ell$ .

We say a function  $f$  is  $\#P$ -hard with respect to weighted reductions if any function in  $\#P$  can be reduced to  $f$  via a weighted reduction.

Having decided how to define hardness, we must look at membership. As counting problems generate integers, we cannot really say that probabilistic inference problems belong to any class in Valiant’s counting hierarchy. In fact, in his seminal work on the complexity of Bayesian networks, Roth

notes that “strictly speaking the problem of computing the degree of belief is not in  $\#P$ , but easily seem equivalent to a problem in this class” [115]. The challenge is to formalize such an equivalence.

Grove, Halpern, and Koller quantify the complexity of probabilistic inference by allowing polynomial computations to occur after counting [57, Definition 4.12]. Their strategy is to say that  $f$  is  $\#P$ -easy if there exists  $f' \in \#P$  and  $f'' \in FP$  such that for all  $\ell$  we have  $f(\ell) = f''(f'(\ell))$ . Similarly, Campos, Stamoulis and Weyland take  $f$  to be  $\#P[1]$ -equivalent if  $f$  is  $\#P$ -hard (in Valiant’s sense) and belongs to  $FP^{\#P[1]}$ . Here the superscript  $\#P[1]$  means that the oracle  $\#P$  can be called only once. It is certainly a good idea to resort to a new term (“equivalence”) in this context; however one must feel that membership to  $FP^{\#P[1]}$  is too weak a requirement given Toda and Watanabe’s theorem [128]: any function in  $\#PH$  can be produced within  $FP^{\#P[1]}$ .

We adopt a stronger notion of equivalence: a function  $f$  is  $\#P$ -equivalent if it is  $\#P$ -hard with respect to weighted reductions and  $g \cdot f$  is in  $\#P$  for some polynomial-time function  $g$  from the input language to rational numbers.

Also, we need to define a class of functions that corresponds to the complexity class PEXP. We might extend Valiant’s definitions and take  $\#EXP$  to be  $\cup_{L \in EXP} FP^L$ . However functions in such a class produce numbers whose size is at most polynomial on the size of the input, as the number of accepting paths of a nondeterministic Turing machine on input  $\ell$  is bounded by  $2^{p(|\ell|)}$  where  $p$  is polynomial and  $|\ell|$  is the length of  $\ell$ . This is not appropriate for our purposes, as even simple specifications may lead to exponentially long output (for instance, take  $\mathbb{P}(X(\chi) = 1) = 1/2$  and compute  $\mathbb{P}(\exists \chi : X(\chi))$ : we must be able to write the answer  $1 - 1/2^N$  using  $N$  bits, and this number of bits is exponential on the input if the domain size is given in binary notation). For this reason, we take  $\#EXP$  to be the class of functions that can be computed by counting machines of exponential time complexity [97]. We say that a function  $f$  is  $\#EXP$ -equivalent if  $f$  is  $\#EXP$ -hard with respect to reductions that follow exactly Definition 7, except for the fact that  $g_1$  may be an exponential time function, and  $gf$  is in  $\#EXP$  for some exponential time function  $g$  from the input language to the rational numbers.

Now consider polynomial bounds on space. We will use the class  $\sharp PSPACE$  class defined by Ladner [77], consisting of those functions that can be computed by counting Turing machines with a polynomial space bound *and* a polynomial bound on the number of nondeterministic moves. This class is actually equal to  $FPSPACE[poly]$ , the class of functions computable in polynomial space whose outputs are strings encoding numbers in binary notation, and bounded in length by a polynomial [77, Theorem 2]. We say that a function is  $\sharp PSPACE$ -equivalent if  $f$  is  $\sharp PSPACE$ -hard with respect to weighted reductions (as in Definition 7), and  $g \cdot f$  is in  $\sharp PSPACE$  for some polynomial space function  $g$  from the input language to the rational numbers. Of course we might have used “ $FPSPACE[poly]$ -equivalent” instead, but we have decided to follow Ladner’s original notation.

There is one more word of caution when it comes to adopting Valiant’s counting Turing machines. It is actually likely that functions that are proportional to conditional probabilities  $\mathbb{P}(\mathbf{Q}|\mathbf{E})$  cannot be produced by counting Turing machines, as classes in Valiant’s counting hierarchy are not likely to be closed under division even by polynomial-time computable functions [95]. Thus we must focus on inferences of the form  $\mathbb{P}(\mathbf{Q})$ ; indeed this is the sort of computation that is analyzed in probabilistic databases [123].

The drawback of Valiant’s hierarchy is, therefore, that a significant amount of adaptation is needed before that hierarchy can be applied to probabilistic inference. But after all this preliminary work, we can convert our previous results accordingly. For instance, we have:

**Theorem 20.** *Consider the class of functions that gets as input a relational Bayesian network specification based on FFFO, a domain size  $N$  (in binary or unary notation), and a set of assignments  $\mathbf{Q}$ , and returns  $\mathbb{P}(\mathbf{Q})$ . This class of functions is  $\#EXP$ -equivalent.*

**Theorem 21.** *Consider the class of functions that gets as input a relational Bayesian network specification based on FFFO with relations of bounded arity, a domain size  $N$  in unary notation, and a set of assignments  $\mathbf{Q}$ , and returns  $\mathbb{P}(\mathbf{Q})$ . This class of functions is  $\sharp PSPACE$ -equivalent.*



**Theorem 22.** *Consider the class of functions that gets as input a relational Bayesian network specification based on  $\text{FFFO}^k$  for  $k \geq 2$ , a domain size  $N$  in unary notation, and a set of assignments  $\mathbf{Q}$ , and returns  $\mathbb{P}(\mathbf{Q})$ . This class of functions is  $\#P$ -equivalent.*

**Theorem 23.** *Consider the class of functions that get as input a plate model based on  $\text{FFFO}$ , a domain size  $N$  (either in binary or unary notation), and a set of assignments  $\mathbf{Q}$ , and returns  $\mathbb{P}(\mathbf{Q})$ . This class of functions is  $\#P$ -equivalent.*

## 9 Conclusion

We have presented a framework for specification and analysis of Bayesian networks, particularly networks containing repetitive patterns that can be captured using function-free first-order logic. Our specification framework is based on previous work on probabilistic programming and structural equation models; our analysis is based on notions of complexity (inferential, combined, query, data, domain) that are similar to concepts used in lifted inference and in probabilistic databases.

Our emphasis was on knowledge representation; in particular we wanted to understand how features of the specification language affect the complexity of inferences. Thus we devoted some effort to connect probabilistic modeling with knowledge representation formalisms, particularly description logics. We hope that we have produced here a sensible framework that unifies several disparate efforts, a contribution that may lead to further insight into probabilistic modeling.

Another contribution of this work is a collection of results on complexity of inferences, as summarized by Table 1 and related commentary. We have also introduced relational Bayesian network specifications based on the  $\text{DLLite}^{\text{nf}}$  logic, a language that can be used to specify probabilistic ontologies and a sizeable class of probabilistic entity-relationship diagrams. In proving results about  $\text{DLLite}^{\text{nf}}$ , we have identified a class of model counting problems with tractability guarantees. Finally, we have shown how to transfer our results into plate models and PRMs, and in doing so we have presented a much needed analysis of these popular specification formalisms.

There are several avenues open for future work. Ultimately, we must reach an understanding of the relationship between expressivity and complexity of Bayesian networks specifications that is as rich as the understanding we now have about the expressivity and complexity of logical languages. We must consider Bayesian networks specified by operators from various description and modal logics, or look at languages that allow variables to have more than two values. In a different direction, we must look at parameterized counting classes [45], so as to refine the analysis even further. There are also several problems that go beyond the inferences discussed in this paper: for example, the computation of Maximum a Posteriori (MAP) configurations, and the verification that a possibly cyclic PRM is consistent for every possible skeleton. There are also models that encode structural uncertainty, say about the presence of edges [71], and novel techniques must be developed to investigate the complexity of such models.

## Acknowledgements

The first author is partially supported by CNPq (grant 308433/2014-9) and the second author is supported by FAPESP (grant 2013/23197-4). We thank Cassio Polpo de Campos for valuable insights concerning complexity proofs, and Johan Kwisthout for discussion concerning the MAJSAT problem.

## A Proofs

**Proposition 1.** *Both  $\#3\text{SAT}(>)$  and  $\#(1\text{-in-}3)\text{SAT}(>)$  are PP-complete with respect to many-one reductions.*

$L_1$	$L_2$	$L_3$	$B_1$	$B_2$	$B_3$	$B_4$	$B_5$
0	0	1	0	0	1	0	0
0	1	0	0	0	0	0	1
0	1	1	0	0	0	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	0

Table 2: Assignments that satisfy  $\nu(C)$ .

*Proof.* Consider first  $\#3\text{SAT}(>)$ . It belongs to PP because deciding  $\#\phi > k$ , for propositional sentence  $\phi$ , is in PP [119, Theorem 4.4]. And it is PP-hard because it is already PP-complete when the input is  $k = 2^{n/2} - 1$  [7, Proposition 1].

Now consider  $\#(1\text{-in-}3)\text{SAT}(>)$ . Suppose the input is a propositional sentence  $\phi$  in 3CNF with propositions  $A_1, \dots, A_n$  and  $m$  clauses. Turn  $\phi$  into another sentence  $\varphi$  in 3CNF by turning each clause  $L_1 \vee L_2 \vee L_3$  in  $\phi$  into a set of four clauses:

$$\neg L_1 \vee B_1 \vee B_2, \quad L_2 \vee B_2 \vee B_3, \quad \neg L_3 \vee B_3 \vee B_4, \quad B_1 \vee B_3 \vee B_5,$$

where the  $B_j$  are fresh propositions not in  $\phi$ . We claim that  $\#\varphi = \#(1\text{-in-}3)\phi$ ; that is,  $\#(1\text{-in-}3)\phi > k$  is equivalent to  $\#\phi > k$ , proving the desired hardness.

To prove this claim in the previous sentence, reason as follows. Define  $\theta(L_1, L_2, L_3) = (L_1 \wedge \neg L_2 \wedge \neg L_3) \vee (\neg L_1 \wedge L_2 \wedge \neg L_3) \vee (\neg L_1 \wedge \neg L_2 \wedge L_3)$ ; that is,  $\theta(L_1, L_2, L_3)$  holds if exactly one of its input literals is true. And for a clause  $\rho = (L_1 \vee L_2 \vee L_3)$ , define

$$\nu(\rho) = \theta(\neg L_1, B_1, B_2) \wedge \theta(L_2, B_2, B_3) \wedge \theta(\neg L_3, B_3, B_4) \wedge \theta(B_1, B_3, B_5),$$

where each  $B_j$  is a fresh proposition not in  $\phi$ . Note that for each assignment to  $(L_1, L_2, L_3)$  that satisfies  $\rho$  there is only one assignment to  $(B_1, B_2, B_3, B_4, B_5)$  that satisfies  $\nu(\rho)$ . To prove this, Table 2 presents the set of *all* assignments that satisfy  $\nu(C)$ . Consequently,  $\#\rho = \#\nu(\rho)$ . By repeating this argument for each clause in  $\phi$ , we obtain our claim.  $\square$

**Theorem 1.**  $\text{INF}[\text{Prop}(\wedge)]$  is in to P when the query  $(\mathbf{Q}, \mathbf{E})$  contains only positive assignments, and  $\text{INF}[\text{Prop}(\vee)]$  is in to P when the query contains only negative assignments.

*Proof.* Consider first  $\text{INF}[\text{Prop}(\wedge)]$ . To run inference with positive assignments  $(\mathbf{Q}, \mathbf{E})$ , just run d-separation to collect the set of root variables that must be true given the assignments (note that as soon as a node is set to true, its parents must be true, and so on recursively). Then the probability of the conjunction of assignments in  $\mathbf{Q}$  and in  $\mathbf{E}$  is just the product of probabilities for these latter atomic propositions to be true, and these probabilities are given in the network specification. Thus we obtain  $\mathbb{P}(\mathbf{Q}, \mathbf{E})$ . Now repeat the same polynomial computation only using assignments in  $\mathbf{E}$ , to obtain  $\mathbb{P}(\mathbf{E})$ , and determine whether  $\mathbb{P}(\mathbf{Q}, \mathbf{E}) / \mathbb{P}(\mathbf{E}) > \gamma$  or not.

Now consider  $\text{INF}[\text{Prop}(\vee)]$ . For any input network specification, we can easily build a network specification in  $\text{INF}[\text{Prop}(\wedge)]$  by turning every variable  $X$  into a new variable  $X'$  such that  $X = \neg X'$ . Then the root node associated with assessment  $\mathbb{P}(X = 1) = \alpha$  is turned into a root node associated with  $\mathbb{P}(X' = 1) = 1 - \alpha$ , and a definition axiom  $X \equiv \vee_i Y_i$  is turned into a definition axiom  $X' \equiv \wedge_i Y'_i$ . Any negative evidence is then turned into positive evidence, and the reasoning in the previous paragraph applies.  $\square$

**Theorem 2.**  $\text{INF}[\text{Prop}(\wedge)]$  and  $\text{INF}[\text{Prop}(\vee)]$  are PP-complete with respect to many-one reductions.

*Proof.* Membership follows from the fact that  $\text{INF}[\text{Prop}(\wedge, \neg)] \in \text{PP}$ . We therefore focus on hardness.

Consider first  $\text{INF}[\text{Prop}(\wedge)]$ . We present a parsimonious reduction from  $\#(1\text{-in-3})\text{SAT}(>)$  to  $\text{INF}[\text{Prop}(\wedge)]$ , following a strategy by Mauá et al. [38].

Take a sentence  $\phi$  in 3CNF with propositions  $A_1, \dots, A_n$  and  $m$  clauses. If there is a clause with a repeated literal (for instance,  $(A_1 \vee A_1 \vee A_2)$  or  $(\neg A_1 \vee A_2 \vee \neg A_1)$ ), then there is no assignment respecting the 1-in-3 rule, so the count can be immediately assigned zero. So we assume that no clause contains a repeated literal in the remainder of this proof.

For each literal in  $\phi$ , introduce a random variable  $X_{ij}$ , where  $i$  refers to the  $i$ th clause, and  $j$  refers to the  $j$ th literal (note:  $j \in \{1, 2, 3\}$ ). The set of all such random variables is  $\mathbf{L}$ .

For instance, suppose we have the sentence  $(A_1 \vee A_2 \vee A_3) \wedge (A_4 \vee \neg A_1 \vee A_3)$ . We then make the correspondences:  $X_{11} \rightsquigarrow A_1$ ,  $X_{12} \rightsquigarrow A_2$ ,  $X_{13} \rightsquigarrow A_3$ ,  $X_{21} \rightsquigarrow A_4$ ,  $X_{22} \rightsquigarrow \neg A_1$ ,  $X_{23} \rightsquigarrow A_3$ .

Note that  $\{X_{ij} = 1\}$  indicates an assignment of **true** to the corresponding literal. Say that a configuration of  $\mathbf{L}$  is *gratifying* if  $X_{i1} + X_{i2} + X_{i3} \geq 1$  for every clause (without necessarily respecting the 1-in-3 rule). Say that a configuration is *respectful* if it respects the 1-in-3 rule; that is, if  $X_{i1} + X_{i2} + X_{i3} \leq 1$  for every clause. And say that a configuration is *sensible* if two variables that correspond to the same literal have the same value, and two variables that correspond to a literal and its negation have distinct values (in the example in the last paragraph, both  $\{X_{11} = 1, X_{22} = 1\}$  and  $\{X_{13} = 1, X_{23} = 0\}$  fail to produce a sensible configuration).

For each random variable  $X_{ij}$ , introduce the assessment  $\mathbb{P}(X_{ij} = 1) = 1 - \varepsilon$ , where  $\varepsilon$  is a rational number determined later. Our strategy is to introduce definition axioms so that only the gratifying-respectful-sensible configurations of  $\mathbf{L}$  get high probability, while the remaining configurations have low probability. The main challenge is to do so without negation.

Let  $\mathbf{Q}$  be an initially empty set of assignments. We first eliminate the configurations that do not respect the 1-in-3 rule. To do so, for  $i = 1, \dots, m$  include definition axioms

$$Y_{i12} \equiv X_{i1} \wedge X_{i2}, \quad Y_{i13} \equiv X_{i1} \wedge X_{i3}, \quad Y_{i32} \equiv X_{i2} \wedge X_{i3}, \quad (6)$$

and add  $\{Y_{i12} = 0, Y_{i13} = 0, Y_{i32} = 0\}$  to  $\mathbf{Q}$ . This guarantees that configurations of  $\mathbf{L}$  that fail to be respectful are incompatible with  $\mathbf{Q}$ .

We now eliminate gratifying-respectful configurations that are not sensible. We focus on gratifying and respectful configurations because, as we show later, ungratifying configurations compatible with  $\mathbf{Q}$  are assigned low probability.

- Suppose first that we have clause where the same literal appears twice. For instance, suppose we have  $(A_i \vee \neg A_i \vee L)$ , where  $L$  is a literal. Assume the literals of this clause correspond to variables  $X_{i1}$ ,  $X_{i2}$ , and  $X_{i3}$ . Then impose  $\{X_{i3} = 0\}$ . All other cases where a clause contains a literal and its negation must be treated similarly.
- Now suppose we have two clauses  $(A \vee L_{i2} \vee L_{i3})$  and  $(\neg A \vee L_{j2} \vee L_{j3})$ , where  $A$  is a proposition and the  $L_{uv}$  are literals (possibly referring more than once to the same propositions). Suppose the six literals in these two clauses correspond to variables  $(X_{i1}, X_{i2}, X_{i3})$  and  $(X_{j1}, X_{j2}, X_{j3})$ , in this order. We must have  $X_{i1} = 1 - X_{j1}$ . To encode this relationship, we take two steps. First, introduce the definition axiom

$$Y_{i1j1} \equiv X_{i1} \wedge X_{j1},$$

and add  $\{Y_{i1j1} = 0\}$  to  $\mathbf{Q}$ : at most one of  $X_{i1}$  and  $X_{j1}$  is equal to 1, but there may still be gratifying-respectful configurations where  $X_{i1} = X_{j1} = 0$ . Thus the second step is enforce the sentence  $\theta = \neg(L_{i2} \vee L_{i3}) \vee \neg(L_{j2} \vee L_{j3})$ , as this forbids  $X_{i1} = X_{j1} = 0$ . Note that  $\theta$  is equivalent to  $\neg(L_{i2} \wedge L_{j2}) \wedge \neg(L_{i2} \wedge L_{j3}) \wedge \neg(L_{i3} \wedge L_{j2}) \wedge \neg(L_{i3} \wedge L_{j3})$ , so introduce the definition axiom

$$Y_{iujv} \equiv X_{iu} \wedge X_{jv}$$

and add  $\{Y_{iujv} = 0\}$  to  $\mathbf{Q}$ , for each  $u \in \{2, 3\}$  and  $v \in \{2, 3\}$ . Proceed similarly if the literals of interest appear in other positions in the clauses.

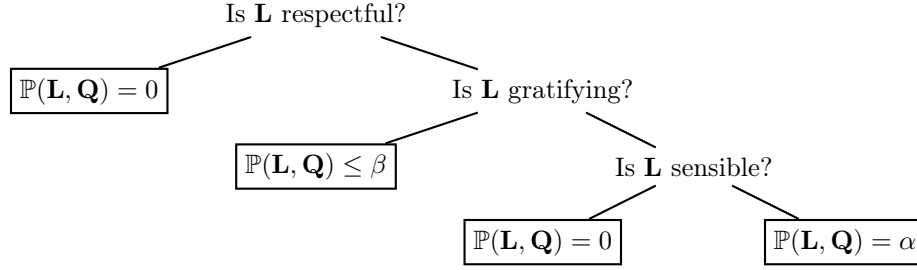


Figure 13: Decision tree of the probability assigned to configurations of the network constructed in the Proof.

- We must now deal with cases where the same literal appears in different positions; recall that no clause contains a repeated literal. So we focus on two clauses that share a literal. Say we have  $(A \vee L_{i2} \vee L_{i3})$  and  $(A \vee L_{j2} \vee L_{j3})$  where the symbols are as in the previous bullet, and where the literals are again paired with variables  $(X_{i1}, X_{i2}, X_{i3})$  and  $(X_{j1}, X_{j2}, X_{j3})$ . If  $X_{i1} = 1$ , then we must have  $X_{j1} = 1$ , and to guarantee this in a gratifying-respectful configuration we introduce

$$Y_{i1j2} \equiv X_{i1} \wedge X_{j2}, \quad Y_{i1j3} \equiv X_{i1} \wedge X_{j3},$$

and add  $\{Y_{i1j2} = 0, Y_{i1j3} = 0\}$  to  $\mathbf{Q}$ . Similarly, if  $X_{j1} = 1$ , we must have  $X_{i1} = 1$ , so introduce

$$Y_{i2j1} \equiv X_{i2} \wedge X_{j1}, \quad Y_{i3j1} \equiv X_{i3} \wedge X_{j1},$$

and add  $\{Y_{i2j1} = 0, Y_{i3j1} = 0\}$  to  $\mathbf{Q}$ . Again, proceed similarly if the literals of interest appear in other positions in the clauses.

Consider a configuration  $x_{11}, \dots, x_{m3}$  of  $\mathbf{L}$ . If this is a gratifying-respectful-sensible configuration, we have that

$$\mathbb{P}(X_{11} = x_{11}, \dots, X_{m3} = x_{m3}) = (1 - \varepsilon)^m \varepsilon^{2m} = \alpha.$$

If the configuration is respectful but *not* gratifying, then

$$\mathbb{P}(X_{11} = x_{11}, \dots, X_{m3} = x_{m3}) \leq (1 - \varepsilon)^{m-1} \varepsilon^{2m+1} = \beta.$$

The number of respectful configurations is at most  $4^m$ , since for each  $i$  there are 4 ways to assign values to  $(X_{i1}, X_{i2}, X_{i3})$  such that  $X_{i1} + X_{i2} + X_{i3} \leq 1$ .

The whole reasoning is illustrated in the decision tree in Figure 13.

If the number of solutions to the original problem is strictly greater than  $k$  then  $\mathbb{P}(\mathbf{Q}) \geq (k+1)\alpha$ . And if the number of solutions is smaller or equal than  $k$  then  $\mathbb{P}(\mathbf{Q}) \leq k\alpha + 4^m\beta$ . Now we must choose  $\varepsilon$  so that  $(k+1)\alpha > k\alpha + 4^m\beta$ , so that we can differentiate between the two cases. We do so by choosing  $\varepsilon < 1/(1 + 4^m)$ . Then  $(\varphi, k)$  is in the language  $\#(1\text{-in-}3)\text{SAT}(>)$  iff  $\mathbb{P}(\mathbf{Q}) > k\alpha$ .

The whole construction is polynomial: the number of definition axioms is at most quadratic in the number of literals of  $\varphi$ , and  $\varepsilon$  can be encoded with  $\mathcal{O}(m+n)$  bits.

Because the construction just described is somewhat complicated, we present an example. Consider again the sentence  $(A_1 \vee A_2 \vee A_3) \wedge (A_4 \vee \neg A_1 \vee A_3)$  and the related variables  $X_{ij}$ . We introduce definitions enforcing the 1-in-3 rule:

$$\begin{array}{lll} Y_{112} \equiv X_{11} \wedge X_{12} & Y_{113} \equiv X_{11} \wedge X_{13} & Y_{123} \equiv X_{12} \wedge X_{13}, \\ Y_{212} \equiv X_{21} \wedge X_{22} & Y_{213} \equiv X_{21} \wedge X_{23} & Y_{223} \equiv X_{22} \wedge X_{23}. \end{array}$$

and appropriate assignments in  $\mathbf{Q}$ . We then guarantee that at most one of  $A_1$  and  $\neg A_1$  is true, by introducing  $Y_{1122} \equiv X_{11} \wedge X_{22}$ , and by adding  $\{Y_{1122} = 0\}$  to  $\mathbf{Q}$ . Note that are configurations that

are not sensible but that satisfy the previous constraints: for instance,  $\{L_{13} = L_{23} = 1, L_{11} = L_{12} = L_{21} = L_{22} = 0\}$  is not sensible and has probability  $\alpha = (1 - \varepsilon)^2 \varepsilon^4$ . To remove those configurations that are not sensible but that have “high” probability, we introduce:

$$\begin{aligned} Y_{1221} &\equiv X_{12} \wedge X_{21}, & Y_{1223} &\equiv X_{12} \wedge X_{23}, \\ Y_{1321} &\equiv X_{13} \wedge X_{21}, & Y_{1323} &\equiv X_{13} \wedge X_{23}, \\ Y_{1322} &\equiv X_{13} \wedge X_{22}, & Y_{1322} &\equiv X_{13} \wedge X_{22}, \\ Y_{1123} &\equiv X_{11} \wedge X_{23}, & Y_{1223} &\equiv X_{12} \wedge X_{23}, \end{aligned}$$

and we add  $\{E_{1221} = 0, E_{1223} = 0, E_{1321} = 0, E_{1323} = 0, E_{1321} = 0, E_{1322} = 0, E_{1123} = 0, E_{1223} = 0\}$  to  $\mathbf{Q}$ . There are  $2^6 = 64$  configurations of  $X_{11}, \dots, X_{23}$ , and 15 of them have  $X_{i1} = X_{i2} = X_{i3} = 0$  for  $i = 1$  or  $i = 2$  (or both). Among these ungratifying configurations, 8 do not respect the 1-in-3 rule; the remaining 7 that respect the 1-in-3 rule are assigned at most probability  $\beta$ . Among the 49 gratifying configurations (i.e., those that assign  $X_{ij} = 1$  for some  $j$  for  $i = 1, 2$ ), 40 do not respect the 1-in-3 rule. Of the remaining 9 configurations, 7 are not sensible. The last 2 configurations are assigned probability  $\alpha$ . We thus have that

$$\mathbb{P}(\mathbf{Q}) = \sum_{x_{11}, \dots, x_{23}} \mathbb{P}(X_{11} = x_{11}, \dots, X_{23} = x_{23}, \mathbf{Q}) \leq 2\alpha + 7\beta,$$

which implies that  $(\varphi, 3)$  is not in  $\#(1\text{-in-}3)\text{SAT}(>)$ ; indeed, there are  $2 < 3$  assignments to  $A_1, A_2, A_3, A_4$  that satisfy  $\varphi$  and respect the 1-in-3 rule.

This concludes our discussion of  $\text{INF}[\text{Prop}(\wedge)]$ , so we move to  $\text{INF}[\text{Prop}(\vee)]$ . To prove its PP-completeness, we must do almost exactly the same construction described before, with a few changes that we enumerate.

First, we associate each literal with a random variable  $X_{ij}$  as before, but now  $X_{ij}$  stands for a *negated* literal. That is, if the literal corresponding to  $X_{ij}$  is  $A$  and  $A$  is *true*, then  $\{X_{ij} = 0\}$ . Thus we must associate each  $X_{ij}$  with the assessment  $\mathbb{P}(X_{ij} = 1) = \varepsilon$ . Definitions must change accordingly: a configuration is now gratifying if  $X_{i1} + X_{i2} + X_{i3} < 3$ .

Second, the previous construction used a number of definition axioms of the form

$$Y \equiv X \wedge X',$$

with associated assignment  $\{Y = 0\}$ . We must replace each such pair by a definition axiom

$$Y \equiv X \vee X'$$

and an assignment  $\{Y = 1\}$ ; recall that  $X$  is just the negation of the variable used in the previous construction, so the overall effect of the constraints is the same.

All other arguments carry, so we obtain the desired hardness.  $\square$

It is instructive to look at a proof of Theorem 2 that uses Turing reductions, as it is much shorter than the previous proof:

*Proof.* To prove hardness of  $\text{INF}[\text{Prop}(\vee)]$ , we use the fact that the function  $\#\text{MON2SAT}$  is  $\#\text{P}$ -complete with respect to Turing reductions [131, Theorem 1]. Recall that  $\#\text{MON2SAT}$  is the function that counts the number of satisfying assignments of a monotone sentence in 2CNF.

So, we can take any MAJSAT problem where the input is sentence  $\phi$  and produce (with polynomial effort) another sentence  $\phi'$  in 2CNF such that  $\#\phi$  is obtained from  $\#\phi'$  (again with polynomial effort). And we can compute  $\#\phi'$  using  $\text{INF}[\text{Prop}(\vee)]$ , as follows. Write  $\phi'$  as  $\bigwedge_{i=1}^m (A_{i1} \vee A_{i2})$ , where each  $A_{ij}$  is a proposition in  $A_1, \dots, A_n$ . Introduce fresh propositions/variables  $C_i$  and definition axioms  $C_i \equiv A_{i1} \vee A_{i2}$ . Also introduce  $\mathbb{P}(A_i = 1) = 1/2$  for each  $A_i$ , and consider the query  $\mathbf{Q} = \{C_1, \dots, C_m\}$ . Note that  $\mathbb{P}(\mathbf{Q}) > \gamma$  if and only if  $\#\phi' = 2^n \mathbb{P}(\mathbf{Q}) > 2^n \gamma$ , so we can bracket  $\#\phi'$ . From  $\#\phi'$  we obtain  $\#\phi$  and we can decide whether  $\#\phi > 2^{n-1}$ , thus solving the original MAJSAT problem.

$\sigma^0$	...	...	...	...	$\sigma^{i-1}$	$(q_a \sigma^i)$	$\sigma^{i+1}$	...	row $2^n - 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	repeat
$\sigma^0$	...	...	...	...	$\sigma^{i-1}$	$(q_a \sigma^i)$	$\sigma^{i+1}$	...	repeat
$\sigma^0$	...	...	...	...	$\sigma^{i-1}$	$(q_a \sigma^i)$	$\sigma^{i+1}$	...	acceptance
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	computations
$(q_0 \sigma_*^0)$	$\sigma_*^1$	...	$\sigma_*^{m-1}$	$\sqcup$	...	...	...	$\sqcup$	row 0 (input)

Figure 14: An accepting computation.

To prove hardness of  $\text{INF}[\text{Prop}(\wedge)]$ , note that the number of satisfying assignments of  $\phi'$  in 2CNF is equal to the number of satisfying assignments of  $\bigwedge_{i=1}^m (\neg A_{i_1} \vee \neg A_{i_2})$ , because one can take each satisfying assignment for the latter sentence and create a satisfying assignment for  $\phi'$  by interchanging true and false, and likewise for the unsatisfying assignments. Introduce fresh propositions/variables  $C_i$  and definition axioms  $C_i \equiv A_{i_1} \wedge A_{i_2}$ . Also introduce  $\mathbb{P}(A_i = 1) = 1/2$  for each  $A_i$ , and consider the query where  $\mathbf{Q} = \{\neg C_1, \dots, \neg C_m\}$ . Again we can bracket the number of assignments that satisfy  $\phi'$ , and thus we can solve any MAJSAT problem by using  $\text{INF}[\text{Prop}(\wedge)]$  and appropriate auxiliary polynomial computations.  $\square$

**Theorem 4.**  $\text{INF}[\text{FFFO}]$  is PEXP-complete with respect to many-one reductions, regardless of whether the domain is specified in unary or binary notation.

*Proof.* To prove membership, note that a relational Bayesian network specification based on FFFO can be grounded into an exponentially large Bayesian network, and inference can be carried out in that network using a counting Turing machine with an exponential bound on time. This is true even if we have unbounded arity of relations: even if we have domain size  $2^N$  and maximum arity  $k$ , grounding each relation generates up to  $2^{kN}$  nodes, still an exponential quantity in the input.

To prove hardness, we focus on binary domain size  $N$  as this simplifies the notation. Clearly if  $N$  is given in unary, then an exponential number of groundings can be produced by increasing the arity of relations (even if the domain is of size 2, an arity  $k$  leads to  $2^k$  groundings).

Given the lack of PEXP-complete problems in the literature, we have to work directly from Turing machines. Start by taking any language  $\mathcal{L}$  such that  $\ell \in \mathcal{L}$  if and only if  $\ell$  is accepted by more than half of the computation paths of a nondeterministic Turing machine  $\mathbb{M}$  within time  $2^{p(|\ell|)}$  where  $p$  is a polynomial and  $|\ell|$  denotes the size of  $\ell$ . To simplify matters, denote  $p(|\ell|)$  by  $n$ . The Turing machine is defined by its alphabet, its states, and its transition function.

Denote by  $\sigma$  a symbol in  $\mathbb{M}$ 's alphabet, and by  $q$  a state of  $\mathbb{M}$ . A configuration of  $\mathbb{M}$  can be described by a string  $\sigma^0 \sigma^1 \dots \sigma^{i-1} (q \sigma^i) \sigma^{i+1} \dots \sigma^{2^n-1}$ , where each  $\sigma^j$  is a symbol in the tape,  $(q \sigma^i)$  indicates both the state  $q$  and the position of the head at cell  $i$  with symbol  $\sigma^i$ . The initial configuration is  $(q_0 \sigma^0) \sigma_*^1 \dots \sigma_*^{m-1}$  followed by  $2^n - m$  blanks, where  $q_0$  is the initial state. There are also states  $q_a$  and  $q_r$  that respectively indicate acceptance or rejection of the input string  $\sigma_*^0 \dots \sigma_*^{m-1}$ . We assume that if  $q_a$  or  $q_r$  appear in some configuration, then the configuration is not modified anymore (that is, the transition moves from this configuration to itself). This is necessary to guarantee that the number of accepting computations is equal to the number of ways in which we can fill in a matrix of computation. For instance, a particular accepting computation could be depicted as a  $2^n \times 2^n$  matrix as in Figure 14, where  $\sqcup$  denotes the blank, and where we complete the rows of the matrix after the acceptance by repeating the accepting row.

The transition function  $\delta$  of  $\mathbb{M}$  takes a pair  $(q, \sigma)$  consisting of a state and a symbol in the machine's tape, and returns a triple  $(q', \sigma', m)$ : the next state  $q'$ , the symbol  $\sigma'$  to be written in the tape (we assume that a blank is never written by the machine), and an integer  $m$  in  $\{-1, 0, 1\}$ . Here  $-1$  means that the head is to move left,  $0$  means that the head is to stay in the current cell, and  $1$  means that the head is to move right.

We now encode this Turing machine using monadic logic, mixing some ideas by Lewis [79] and by Tobies [126].

Take a domain of size  $2^{2n}$ . The idea is that each  $\chi$  is a cell in the computation matrix. From now on, a “point” is a cell in that matrix. Introduce parvariables  $X_0(\chi), \dots, X_{n-1}(\chi)$  and  $Y_0(\chi), \dots, Y_{n-1}(\chi)$  to encode the index of the column and the row of point  $\chi$ . Impose, for  $0 \leq i \leq n-1$ , the assessments  $\mathbb{P}(X_i(\chi) = 1) = \mathbb{P}(Y_i(\chi) = 1) = 1/2$ .

We need to specify the concept of adjacent points in the computation matrix. To this end we introduce two macros, **EAST**( $\chi, y$ ) and **NORTH**( $\chi, y$ ) (note that we do not actually need binary relations here; these expressions are just syntactic sugar). The meaning of **EAST**( $\chi, y$ ) is that for point  $\chi$  there is a point  $y$  that is immediately to the right of  $\chi$ . And the meaning of **NORTH**( $\chi, y$ ) is that for point  $\chi$  there is a point  $y$  that is immediately on top of  $\chi$ .

$$\begin{aligned} \text{EAST}(\chi, y) &:= \bigwedge_{k=0}^{n-1} (\bigwedge_{j=0}^{k-1} X_j(\chi)) \rightarrow (X_k(\chi) \leftrightarrow \neg X_k(y)) \\ &\quad \wedge \bigwedge_{k=0}^{n-1} (\bigvee_{j=0}^{k-1} \neg X_j(\chi)) \rightarrow (X_k(\chi) \leftrightarrow X_k(y)) \\ &\quad \wedge \bigwedge_{k=0}^{n-1} (Y_k(\chi) \leftrightarrow Y_k(y)). \\ \text{NORTH}(\chi, y) &:= \bigwedge_{k=0}^{n-1} (\bigwedge_{j=0}^{k-1} Y_j(\chi)) \rightarrow (Y_k(\chi) \leftrightarrow \neg Y_k(y)) \\ &\quad \wedge \bigwedge_{k=0}^{n-1} (\bigvee_{j=0}^{k-1} \neg Y_j(\chi)) \rightarrow (Y_k(\chi) \leftrightarrow Y_k(y)) \\ &\quad \wedge \bigwedge_{k=0}^{n-1} (X_k(\chi) \leftrightarrow X_k(y)). \end{aligned}$$

Now introduce

$$\begin{aligned} Z_1 &\equiv (\forall \chi : \exists y : \text{EAST}(\chi, y)) \wedge (\forall \chi : \exists y : \text{NORTH}(\chi, y)), \\ Z_2 &\equiv \exists \chi : \bigwedge_{k=0}^{n-1} (\neg X_k(\chi) \wedge \neg Y_k(\chi)). \end{aligned}$$

Now if  $Z_1 \wedge Z_2$  is true, we “build” a square “board” of size  $2^n \times 2^n$  (in fact this is a torus as the top row is followed by the bottom row, and the rightmost column is followed by the leftmost column).

Introduce a relation  $C_j$  for each triplet  $(\alpha, \beta, \gamma)$  where each element of the triplet is either a symbol  $\sigma$  or a symbol of the form  $(q\sigma)$  for our machine  $\mathbb{M}$ , and with an additional condition: if  $(\alpha, \beta, \gamma)$  has  $\beta$  equal to a blank, then  $\gamma$  is a blank. Furthermore, introduce a relation  $C_j$  for each triple  $(\diamond, \beta, \gamma)$ , where  $\beta$  and  $\gamma$  are as before, and  $\diamond$  is a new special symbol (these relations are needed later to encode the “left border” of the board). We refer to each  $C_j$  as a *tile*, as we are in effect encoding a domino system [79]. For each tile, impose  $\mathbb{P}(C_j(\chi) = 1) = 1/2$ .

Now each point must have one and only one tile:

$$Z_3 \equiv \forall \chi : \left( \bigvee_{j=0}^{c-1} C_j(\chi) \right) \wedge \left( \bigwedge_{0 \leq j \leq c-1, 0 \leq k \leq c-1, j \neq k} \neg (C_j(\chi) \wedge C_k(\chi)) \right).$$

Having defined the tiles, we now define a pair of relations encoding the “horizontal” and “vertical” constraints on tiles, so as to encode the transition function of the Turing machine. Denote by  $H$

the relation consisting of pairs of tiles that satisfy the horizontal constraints and by  $V$  the relation consisting of pairs of tiles that satisfy the vertical constraints.

The horizontal constraints must enforce the fact that, in a fixed row, a tile  $(\alpha, \beta, \gamma)$  at column  $i$  for  $0 \leq i \leq 2^n - 1$  overlaps the tile  $(\alpha', \beta', \gamma')$  at column  $i + 1$  by satisfying

$$((\alpha, \beta, \gamma), (\alpha', \beta', \gamma')) : \beta = \alpha', \gamma = \beta'.$$

The vertical constraints must encode the possible computations. To do so, consider a tile  $t = (\alpha, \beta, \gamma)$  at row  $j$ , for  $0 \leq j \leq 2^n - 1$ , and tile  $t' = (\alpha', \beta', \gamma')$  at row  $j + 1$ , both at the same column. The pair  $(t, t')$  is in  $V$  if and only if (a)  $t'$  can be reached from  $t$  given the states in the Turing machine; and (b) if  $t = (\diamond, \beta, \gamma)$ , then  $t' = (\diamond, \beta', \gamma')$  for  $\beta'$  and  $\gamma'$  that follow from the behavior of  $\mathbb{M}$ .

We distinguish the last row and the last column, as the transition function does not apply to them:

$$D_X(\chi) \equiv \bigwedge_{k=0}^{n-1} X_k(\chi), \quad D_Y(\chi) \equiv \bigwedge_{k=0}^{n-1} Y_k(\chi).$$

We can now encode the transition function:

$$Z_4 \equiv \forall \chi : \neg D_X(\chi) \rightarrow \left( \bigwedge_{j=0}^{c-1} C_j(\chi) \rightarrow (\forall y : \text{EAST}(\chi, y) \rightarrow \bigvee_{k:(j,k) \in H} C_k(y)) \right),$$

$$Z_5 \equiv \forall \chi : \neg D_Y(\chi) \rightarrow \left( \bigwedge_{j=0}^{c-1} C_j(\chi) \rightarrow (\forall y : \text{NORTH}(\chi, y) \rightarrow \bigvee_{k:(j,k) \in V} C_k(y)) \right).$$

We create a parvariable that signals the accepting state:

$$Z_6 \equiv \exists \chi : \bigvee_{j: C_j \text{ contains } q_a} C_j(\chi).$$

Finally, we must also impose the initial conditions. Take the tiles in the first row so that symbols in the input of  $\mathbb{M}$  are encoded as  $m$  tiles, with the first tile  $t^0 = (\diamond, (q_0 \sigma_*^0), \sigma_*^1)$  and the following ones  $t^j = (\sigma_*^{j-1}, \sigma_*^j, \sigma_*^{j+1})$  up to  $t^{m-1} = (\sigma_*^{m-2}, \sigma_*^{m-1}, \sqcup)$ . So the next tile will be  $(\sigma_*^{m-1}, \sqcup, \sqcup)$ , and after that all tiles in the first row will contain only blanks. Now take individuals  $a_i$  for  $i \in \{0, \dots, m-1\}$  and create an assignment  $\{C_i^0(a_i) = 1\}$  for each  $a_i$ , where  $C_i^0$  is the  $i$ th tile encoding the initial conditions. Denote by  $\mathbf{E}$  the set of such assignments.

Now  $\mathbb{P}(Z_6 | \mathbf{E} \wedge \bigwedge_{i=1}^6 Z_i) > 1/2$  if and only if the number of correct arrangements of tiles that contain the accepting state is larger than the total number of possible valid arrangements. Hence an inference with the constructed relational Bayesian network specification decides the language  $\mathcal{L}$  we started with, as desired.  $\square$

**Theorem 5.**  $\text{INF}[\text{ALC}]$  is PEXP-complete with respect to many-one reductions, when domain size is given in binary notation.

*Proof.* Membership follows from Theorem 4. To prove hardness, consider that by imposing an assessment  $\mathbb{P}(X(\chi, y)) = 1$ , we transform  $\exists y : X(\chi, y) \wedge Y(y)$  into  $\exists y : Y(y)$ . This is all we need (together with Boolean operators) to build the proof of PEXP-completeness in Theorem 4. (The inferential complexity of ALC has been derived, with a different proof, by Cozman and Polastro [30].)  $\square$

**Theorem 6.**  $\text{QINF}[\text{FFFO}]$  is PEXP-complete with respect to many-one reductions, when the domain is specified in binary notation.



*Proof.* Membership is obvious as the inferential complexity is already in PEXP. To show hardness, take a Turing machine  $\mathbb{M}$  that solves some PEXP-complete problem within  $2^n$  steps. That is, there is a PEXP-complete language  $\mathcal{L}$  such that  $\ell \in \mathcal{L}$  if and only if the input string  $\ell$  is accepted by more than half of the computation paths of  $\mathbb{M}$  within time  $2^n$ .

Such a Turing machine  $\mathbb{M}$  has alphabet, states and transitions as in the proof of Theorem 4. Assume that  $\mathbb{M}$  repeats its configuration as soon as it enters into the accepting or the rejecting state, as in the proof of Theorem 4.

To encode  $\mathbb{M}$  we resort to a construction by Gradel [55] where relations of arity two are used. We use: (a) for each state  $q$  of  $\mathbb{M}$ , a unary relation  $X_q$ ; (b) for each symbol  $\sigma$  in the alphabet of  $\mathbb{M}$ , a binary relation  $Y_\sigma$ ; (c) a binary relation  $Z$ . The idea is that  $X_q(\chi)$  means that  $\mathbb{M}$  is in state  $q$  at computation step  $\chi$ , while  $Y_\sigma(\chi, y)$  means that  $\sigma$  is the symbol at the  $y$ th position in the tape at computation step  $\chi$ , and  $Z(\chi, y)$  means that the machine head is at the  $y$ th position in the tape at computation step  $\chi$ . Impose  $\mathbb{P}(X_q(\chi) = 1) = \mathbb{P}(Y_\sigma(\chi, y) = 1) = \mathbb{P}(Z(\chi, y) = 1) = 1/2$ .

We use a distinguished relation  $<$ , assumed not to be in the vocabulary. This relation is to be a linear order on the domain; to obtain this behavior, just introduce  $\mathbb{P}(<(\chi, y) = 1) = 1/2$  and

$$\begin{aligned} Z_1 \equiv & (\forall \chi : \neg(\chi < \chi)) \wedge \\ & (\forall \chi : \forall y : \forall z : (\chi < y \wedge y < z) \rightarrow \chi < z) \wedge \\ & (\forall \chi : \forall y : (\chi < y) \vee (y < \chi) \vee (\chi = y)). \end{aligned}$$

We will later set evidence on  $Z_1$  to force  $<$  to be a linear order. The important point is that we can assume that a domain of size  $2^n$  is given and all elements of this domain are ordered according to  $<$ .

Clearly we can define a successor relation using  $<$ :

$$\text{successor}(\chi, y) \equiv (\chi < y) \wedge (\neg \exists z : (\chi < z) \wedge (z < y)).$$

Also, we can define a relation that signals the “first” individual:

$$\text{first}(\chi) \equiv \neg \exists y : y < \chi.$$

We must guarantee that at any given step the machine is in a single state, each cell of the tape has a single symbol, and the head is at a single position of the tape:

$$\begin{aligned} Z_2 & \equiv \forall \chi : \bigvee_q \left( X_q(\chi) \wedge \bigwedge_{q' \neq q} \neg X_{q'}(\chi) \right), \\ Z_3 & \equiv \forall \chi : \forall y : \bigvee_\sigma \left( Y_\sigma(\chi, y) \wedge \bigwedge_{\sigma' \neq \sigma} \neg Y_{\sigma'}(\chi, y) \right), \\ Z_4 & \equiv \forall \chi : (\exists y : Z(\chi, y) \wedge \forall z : (z \neq y) \rightarrow \neg Z(\chi, z)). \end{aligned}$$

We also have to guarantee that computations do not change the content of a cell that is not visited by the head:

$$Z_5 \equiv \forall \chi : \forall y : \forall z : \bigwedge_\sigma (\neg Z(\chi, y) \wedge Y_\sigma(\chi, y) \wedge \text{successor}(\chi, z)) \rightarrow Y_\sigma(z, y).$$

We must encode the changes made by the transition function:

$$\begin{aligned}
Z_6 \equiv & \forall \chi : \forall y : \forall z : \bigwedge_{q, \sigma} (Z(\chi, y) \wedge Y_\sigma(\chi, y) \wedge X_q(\chi) \wedge \text{successor}(\chi, z)) \rightarrow \\
& \bigvee_{(q' \sigma', 1) \in \delta(q, \sigma)} (X_{q'}(z) \wedge Y_{\sigma'}(z, y) \wedge (\forall w : \text{successor}(y, w) \rightarrow Z(z, w))) \\
& \vee \bigvee_{(q' \sigma', 0) \in \delta(q, \sigma)} (X_{q'}(z) \wedge Y_{\sigma'}(z, y) \wedge Z(z, y)) \\
& \vee \bigvee_{(q' \sigma', 1) \in \delta(q, \sigma)} (X_{q'}(z) \wedge Y_{\sigma'}(z, y) \wedge (\forall w : \text{successor}(w, y) \rightarrow Z(z, w))).
\end{aligned}$$

We must also guarantee that all cells to the right of a blank cell are also blank:

$$Z_7 \equiv \forall \chi : \forall y : \forall z : Y_\sqcup(\chi, y) \wedge \text{successor}(y, z) \rightarrow Y_\sqcup(\chi, z).$$

Finally, we must signal the accepting state:

$$Z_8 \equiv \exists \chi : X_{q_a}(\chi).$$

We have thus created a set of formulas that encode the behavior of the Turing machine. Now take the input string  $\ell$ , equal to  $\sigma_*^0, \sigma_*^1, \dots, \sigma_*^{m-1}$ , and encode it as a query as follows. Start by “creating” the first individual in the ordering by taking the assignment  $\{\text{first}(a_0) = 1\}$ . Then introduce  $\{Z(a_0, a_0) = 1\}$  to initialize the head. Introduce  $\{Y_{\sigma_*^0}(a_0, a_0) = 1\}$  to impose the initial condition on the first cell, and for each subsequent initial condition  $\sigma_*^i$  we set  $\{Y_{\sigma_*^i}(a_0, a_i) = 1\}$  and  $\{\text{successor}(a_{i-1}, a_i) = 1\}$  where  $a_i$  is a fresh individual. Finally, set  $\{Y_\sqcup(a_0, a_m) = 1\}$  and  $\{\text{successor}(a_{m-1}, a_m) = 1\}$  and  $\{X_{q_0}(a_0) = 1\}$ . These assignments are denoted by  $\mathbf{E}$ .

Now  $\mathbb{P}(Z_8 | \mathbf{E} \wedge \bigwedge_{i=1}^8 Z_i) > 1/2$  for a domain of size  $2^n$  if and only if the number of interpretations reaching the accepting state is larger than the total number of possible interpretations encoding computation paths.  $\square$

**Theorem 7.** QINF[FFFO] is PP-complete with respect to many-one reductions when the domain is specified in unary notation.

*Proof.* To prove hardness, take a MAJSAT problem where  $\phi$  is in CNF with  $m$  clauses and propositions  $A_1, \dots, A_n$ . Make sure each clause has at most  $n$  literals by removing repeated literals, or by removing clauses with a proposition and its negation). Make sure  $m = n$ : if  $m < n$ , then add trivial clauses such as  $A_1 \vee \neg A_1$ ; if instead  $n < m$ , then add fresh propositions  $A_{n+1}, \dots, A_m$ . These changes do not change the output of MAJSAT. Introduce unary relations  $\text{sat}(\chi)$  and impose  $\mathbb{P}(\text{sat}(\chi)) = 1/2$ . Take a domain  $\{1, \dots, n\}$ ; the elements of the domain serve a dual purpose, indexing both propositions and clauses. Introduce relations  $\text{sat}(\chi)$ ,  $\text{positiveLit}(\chi, y)$  and  $\text{negativeLit}(\chi, y)$ , assessments  $\mathbb{P}(\text{sat}(\chi) = 1) = \mathbb{P}(\text{positiveLit}(\chi, y) = 1) = \mathbb{P}(\text{negativeLit}(\chi, y) = 1) = 1/2$ , and definition axioms:

$$\begin{aligned}
\text{clause}(\chi) & \equiv \exists y : (\text{positiveLit}(\chi, y) \wedge \text{sat}(y)) \vee (\text{negativeLit}(\chi, y) \wedge \neg \text{sat}(y)), \\
\text{query} & \equiv \forall \chi : \text{clause}(\chi).
\end{aligned}$$

Take evidence  $\mathbf{E}$  as follows. For each clause, run over the literals. Consider the  $i$ th clause, and its non-negated literal  $A_j$ : set  $\text{positiveLit}(i, j)$  to true. And consider negated literal  $\neg A_j$ : set  $\text{negativeLit}(i, j)$  to true. Set all other groundings of  $\text{positiveLit}$  and  $\text{negativeLit}$  to false. Note that  $\mathbb{P}(\mathbf{E}) = 2^{-2n^2} > 0$ . Now decide whether  $\mathbb{P}(\text{query} = 1 | \mathbf{E}) > 1/2$ . If YES, the MAJSAT problem is accepted, if NO, it is not accepted. Hence we have the desired polynomial reduction (the query is quadratic on domain size; all other elements are linear on domain size).

To prove membership in PP, we describe a Turing machine  $\mathbb{M}$  that decides whether  $\mathbb{P}(Q | \mathbf{E}) > \gamma$ . The machine guesses a truth assignment for each one of the polynomially-many grounded root nodes

(and writes the guess in the working tape). Note that each grounded root node  $X$  is associated with an assessment  $\mathbb{P}(X = 1) = c/d$ , where  $c$  and  $d$  are the smallest such integers. Then the machine replicates its computation paths out of the guess on  $X$ : there are  $c$  paths with identical behavior for guess  $\{X = 1\}$ , and  $d - c$  paths with identical behavior for guess  $\{X = 0\}$ .

Now the machine verifies whether the set of guessed truth assignment satisfies  $\mathbf{E}$ ; if not, move to state  $q_1$ . If yes, then verify whether the guessed truth assignment fails to satisfy  $\mathbf{Q}$ ; if not, move to state  $q_2$ . And if yes, then move to state  $q_3$ . The key point is that there is a logarithmic space, hence polynomial time, algorithm that can verify whether a set of assignments holds once the root nodes are set [80, Section 6.2].

Suppose that out of  $N$  computation paths that  $\mathbb{M}$  can take,  $N_1$  of them reach  $q_1$ ,  $N_2$  reach  $q_2$ , and  $N_3$  reach  $q_3$ . By construction,

$$N_1/N = 1 - \mathbb{P}(\mathbf{E}), \quad N_2/N = \mathbb{P}(\neg\mathbf{Q}, \mathbf{E}), \quad N_3/N = \mathbb{P}(\mathbf{Q}, \mathbf{E}), \quad (7)$$

where we abuse notation by taking  $\neg\mathbf{Q}$  to mean that some assignment in  $\mathbf{Q}$  is not true. Note that up to this point we do not have any rejecting nor accepting path, so the specification of  $\mathbb{M}$  is not complete.

The remainder of this proof just reproduces a construction by Park in his proof of PP-completeness for propositional Bayesian networks [36]. Park's construction adds rejecting/accepting computation paths emanating from  $q_1$ ,  $q_2$  and  $q_3$ . It uses numbers

$$a = \begin{cases} 1 & \text{if } \gamma < 1/2, \\ 1/(2\gamma) & \text{otherwise,} \end{cases} \quad b = \begin{cases} (1 - 2\gamma)/(2 - 2\gamma) & \text{if } \gamma < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

and the smallest integers  $a_1, a_2, b_1, b_2$  such that  $a = a_1/a_2$  and  $b = b_1/b_2$ . Now, out of  $q_1$  branch into  $a_2b_2$  computation paths that immediately stop at the accepting state, and  $a_2b_2$  computation paths that immediately stop at the rejecting state.<sup>7</sup> Out of  $q_2$  branch into  $2a_2b_1$  paths that immediately stop at the accepting state, and  $2(b_2 - b_1)a_2$  paths that immediately stop at the rejecting state. Out of  $q_3$  branch into  $2a_1b_2$  paths that immediately stop at the accepting state, and  $2(a_2 - a_1)b_2$  paths that immediately stop at the rejecting state. For the whole machine  $\mathbb{M}$ , the number of computation paths that end up at the accepting state is  $a_2b_2N_1 + 2a_2b_1N_2 + 2a_1b_2N_3$ , and the total number of computation paths is  $a_2b_2N_1 + a_2b_2N_1 + 2b_1a_2N_2 + 2(b_2 - b_1)a_2N_2 + 2a_1b_2N_3 + 2(a_2 - a_1)b_2N_3 = 2a_2b_2N$ . Hence the number of accepting paths divided by the total number of paths is  $(N_1(1/2) + (b_1/b_2)N_2 + (a_1/a_2)N_3)/N$ . This ends Park's construction. By combining this construction with Expression (7), we obtain

$$\begin{aligned} \frac{\frac{N_1}{2} + \frac{b_1N_2}{b_2} + \frac{a_1N_3}{a_2}}{N} > 1/2 & \Leftrightarrow \frac{1 - \mathbb{P}(\mathbf{E})}{2} + b\mathbb{P}(\neg\mathbf{Q}, \mathbf{E}) + a\mathbb{P}(\mathbf{Q}, \mathbf{E}) > 1/2 \\ \Leftrightarrow a\mathbb{P}(\mathbf{Q}, \mathbf{E}) + b\mathbb{P}(\neg\mathbf{Q}, \mathbf{E}) > \mathbb{P}(\mathbf{E})/2 & \Leftrightarrow a\mathbb{P}(\mathbf{Q}|\mathbf{E}) + b\mathbb{P}(\neg\mathbf{Q}|\mathbf{E}) > 1/2, \end{aligned}$$

as we can assume that  $\mathbb{P}(\mathbf{E}) > 0$  (otherwise the number of accepting paths is equal to the number of rejecting paths), and then

$$\begin{cases} \text{if } \gamma < 1/2 : & \mathbb{P}(\mathbf{Q}|\mathbf{E}) + \frac{1-2\gamma}{2-2\gamma}(1 - \mathbb{P}(\mathbf{Q}|\mathbf{E})) > 1/2 & \Leftrightarrow \mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma; \\ \text{if } \gamma \geq 1/2 : & (1/(2\gamma))\mathbb{P}(\mathbf{Q}|\mathbf{E}) > 1/2 & \Leftrightarrow \mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma. \end{cases}$$

Hence the number of accepting computation paths of  $\mathbb{M}$  is larger than half the total number of computation paths if and only if  $\mathbb{P}(\mathbf{Q}|\mathbf{E}) > \gamma$ . This completes the proof of membership.  $\square$

**Theorem 8.** *Suppose  $\text{NETIME} \neq \text{ETIME}$ . Then  $\text{DINF}[\text{FFFO}]$  is not solved in deterministic exponential time, when the domain size is given in binary notation.*

<sup>7</sup>The number of created paths may be exponential in the numbers  $a_2$  and  $b_2$ ; however it is always possible to construct a polynomial sequence of steps that encodes an exponential number of paths (say the number of paths has  $B$  bits; then build  $B$  distinct branches, each one of them multiplying alternatives so as to simulate an exponential). This sort of branching scheme is also assumed whenever needed.

*Proof.* Jaeger describes results implying, in case  $\text{NETIME} \neq \text{ETIME}$ , that there is a sentence  $\phi \in \text{FFFO}$  such that the *spectrum* of  $\phi$  cannot be recognized in deterministic exponential time [65]. Recall: the spectrum of a sentence is a set containing each integer  $N$ , in binary notation, such that  $\phi$  has a model whose domain size is  $N$  [55]. So take  $N$  in binary notation, the relational Bayesian network specification  $A \equiv \phi$ , and decide whether  $\mathbb{P}(A) > 0$  for domain size  $N$ ; if yes, then  $N$  is in the spectrum of  $\phi$ .  $\square$

**Theorem 9.**  $\text{DINF}[\text{FFFO}]$  is  $\text{PP}_1$ -complete with respect to many-one reductions, when the domain is given in unary notation.

*Proof.* To prove membership, just consider the Turing machine used in the proof of Theorem 7, now with a fixed query. This is a polynomial-time nondeterministic Turing machine that gets the domain size in unary (that is, as a sequence of 1s) and produces the desired output.

To prove hardness, take a Turing machine with input alphabet consisting of symbol 1, and that solves a  $\text{PP}_1$ -complete problem in  $N^m$  steps for input consisting of  $N$  symbols 1. Take the probabilistic assessment and the definition axioms for *successor*, *first*, and  $Z_1$  as in the proof of Theorem 6. Now introduce relations  $X_q$ ,  $Y_\sigma$  and  $Z$  as in that proof, with the difference that  $\chi$  is substituted for  $m$  logvars  $\chi_i$ , and likewise  $y$  is substituted for  $m$  logvars  $\chi_j$ . For instance, we now have  $Z(\chi_1, \dots, \chi_m, y_1, \dots, y_m)$ . Repeat definition axioms for  $Z_2, \dots, Z_8$  as presented in the proof of Theorem 6, with appropriate changes in the arity of relations. In doing so we have an encoding for the Turing machine where the computation steps are indexed by a vector  $[\chi_1, \dots, \chi_m]$ , and the tape is indexed by a vector  $[y_1, \dots, y_m]$ . The remaining problem is to insert the input. To do so, introduce:

$$\begin{aligned} Z_9 \equiv & \forall \chi : \forall y_1 : \dots \forall y_m : \text{first}(\chi) \rightarrow \\ & \left( \bigwedge_{i \in \{2, \dots, m\}} \text{first}(y_i) \rightarrow Y_1(\overbrace{\chi, \dots, \chi}^{m \text{ logvars}}, y_1, \dots, y_m) \right) \\ & \wedge \left( \neg \bigwedge_{i \in \{2, \dots, m\}} \text{first}(y_i) \rightarrow Y_{\perp}(\overbrace{\chi, \dots, \chi}^{m \text{ logvars}}, y_1, \dots, y_m) \right). \end{aligned}$$

Now  $\mathbb{P}(Z_8 \mid \bigwedge_{i=1}^9 Z_i) > 1/2$  for a domain of size  $N$  if and only if the number of interpretations that set an accepting state to true is larger than half the total number of interpretations encoding computation paths.  $\square$

**Theorem 10.**  $\text{INF}[\text{FFFO}]$  is  $\text{PSPACE}$ -complete with respect to many-one reductions, when relations have bounded arity and the domain size is given in unary notation.

*Proof.* To prove membership, construct a Turing machine that goes over the truth assignments for all of the polynomially-many grounded root nodes. The machine generates an assignment, writes it using polynomial space, and verifies whether  $\mathbf{E}$  can be satisfied: there is a polynomial space algorithm to do this, as we basically need to do model checking in first-order logic [55, Section 3.1.4]. While cycling through truth assignments, keep adding the probabilities of the truth assignments that satisfy  $\mathbf{E}$ . If the resulting probability for  $\mathbf{E}$  is zero, reject; otherwise, again go through every truth assignment of the root nodes, now keeping track of how many of them satisfy  $\{\mathbf{Q}, \mathbf{E}\}$ , and adding the probabilities for these assignments. Then divide the probability of  $\{\mathbf{Q}, \mathbf{E}\}$  by the probability of  $\mathbf{E}$ , and compare the result with the rational number  $\gamma$ .

To show hardness, consider the definition axiom  $Y \equiv Q_1 \chi_1 : \dots Q_n \chi_n : \phi(\chi_1, \dots, \chi_n)$ , where each  $Q_i$  is a quantifier (either  $\forall$  or  $\exists$ ) and  $\phi$  is a quantifier-free formula containing only Boolean operators, a unary relation  $X$ , and logvars  $\chi_1, \dots, \chi_n$ . The relation  $X$  is associated with assessment  $\mathbb{P}(X(\chi) = 1) = 1/2$ . Take domain  $\mathcal{D} = \{0, 1\}$  and evidence  $\mathbf{E} = \{X(0) = 0, X(1) = 1\}$ . Then  $\mathbb{P}(Y = 1 \mid \mathbf{E}) > 1/2$  if and only if  $Q_1 \chi_1 : \dots Q_n \chi_n : \phi(\chi_1, \dots, \chi_n)$  is satisfiable. Deciding the latter satisfiability question is in fact equivalent to deciding the satisfiability of a Quantified Boolean Formula, a  $\text{PSPACE}$ -complete problem [80, Section 6.5].  $\square$

Now consider the bounded variable fragment  $\text{FFFO}^k$ . It is important to notice that if the body of every definition axiom belongs to  $\text{FFFO}^k$  for an integer  $k$ , then all definition axioms together are equivalent to a single formula in  $\text{FFFO}^k$ . Hence results on logical inference for  $\text{FFFO}^k$  can be used to derive inferential, query and domain complexities.

**Theorem 11.**  $\text{INF}[\text{FFFO}^k]$  is PP-complete with respect to many-one reductions, for all  $k \geq 0$ , when the domain size is given in unary notation.

*Proof.* Hardness is trivial: even  $\text{Prop}(\wedge, \neg)$  is PP-hard. To prove membership, use the Turing machine described in the proof of membership in Theorem 7, with a small difference: when it is necessary to check whether  $\mathbf{E}$  (or  $\mathbf{Q} \cup \mathbf{E}$ ) holds given a guessed assignment for root nodes, use the appropriate model checking algorithm [136], as this verification can be done in polynomial time.  $\square$

**Theorem 12.**  $\text{QINF}[\text{FFFO}^k]$  is PP-complete with respect to many-one reductions, for all  $k \geq 2$ , when domain size is given in unary notation.

*Proof.* To prove membership, note that  $\text{QINF}[\text{FFFO}]$  is in PP by Theorem 7. To prove hardness, note that the proof of hardness in Theorem 7 uses only  $\text{FFFO}^2$ .  $\square$

**Theorem 13.**  $\text{DINF}[\text{FFFO}^k]$  is  $\text{PP}_1$ -complete with respect to many-one reductions, for  $k > 2$ , and is in P for  $k \leq 2$ , when the domain size is given in unary notation.

*Proof.* For  $k \leq 2$ , results in the literature show how to count the number of satisfying models of a formula in polynomial time [133, 135].

For  $k > 2$ , membership obtains as in the proof of Theorem 9. Hardness has been in essence proved by Beame et al. [9, Lemmas 3.8, 3.9]. We adapt their arguments, simplifying them by removing the need to enumerate the counting Turing machines. Take a Turing machine  $\mathbb{M}$  that solves a  $\#\text{P}_1$ -complete problem in  $N^m$  steps for an input consisting of  $N$  ones. By padding the input, we can always guarantee that  $\mathbb{M}$  runs in time linear in the input. To show this, consider that for the input sequence with  $N$  ones, we can generate another sequence  $S(N)$  consisting of  $f(N) = (2N + 1)2^{m\lceil \log_2 N \rceil}$  ones. Because  $(2^{1+\log_2 N})^m \geq 2^{m\lceil \log_2 N \rceil}$ , we have  $(2N + 1)2^{mN^m} > f(N)$ , and consequently  $S(N)$  can be generated in polynomial time. Modify  $\mathbb{M}$  so that the new machine: (a) receives  $S(N)$ ; (b) in linear time produces the binary representation of  $S(N)$ , using an auxiliary tape;<sup>8</sup> (c) then discards the trailing zeroes to obtain  $2N + 1$ ; (d) obtains  $N$ ; (e) writes  $N$  ones in its tape; (f) then runs the original computation in  $\mathbb{M}$ . Because  $2^{m\lceil \log_2 N \rceil} \geq N^m$ , we have  $f(N) > N^m$ , and consequently the new machine runs in time that is overall linear in the input size  $f(N)$ , and in space within  $f(N)$ . Suppose, to be concrete, that the new machine runs in time that is smaller than  $Mf(N)$  for some integer  $M$ . We just have to encode this machine in  $\text{FFFO}^3$ , by reproducing a clever construction due to Beame et al. [9].

We use the Turing machine encoding described in the proof of Theorem 8, but instead of using a single relation  $Z(\chi, y)$  to indicate the head position at step  $\chi$ , we use

$$Z^{1,1}(\chi, y), \dots, Z^{M,1}(\chi, y), Z^{1,2}(\chi, y), \dots, Z^{M,2}(\chi, y),$$

with the understanding that for a fixed  $\chi$  we have that  $Z^{i,j}(\chi, y)$  yields the position  $y$  of the head in step  $\chi$  and sub-step  $i$ , either in the main tape (tape 1) or in the auxiliary tape (tape 2). So,  $Z^{1,j}$  is followed by  $Z^{2,j}$  and so on until  $Z^{M,j}$  for a fixed step  $\chi$ . Similarly, we use  $X_q^t(\chi)$ ,  $Y_\sigma^{t,1}(\chi, y)$  and  $Y_\sigma^{t,2}(\chi, y)$  for  $t \in \{1, \dots, M\}$ . Definition axioms must be changed accordingly; for instance, we have

$$Z_2 \equiv \forall \chi : \bigwedge_t \bigvee_q \left( X_q^t(\chi) \wedge \bigwedge_{q' \neq q} \neg X_{q'}^t(\chi) \right),$$

<sup>8</sup>For instance: go from left to right replacing pairs 11 by new symbols  $\clubsuit\heartsuit$ ; if a blank is reached in the middle of such a pair, then add a 1 at the first blank in the auxiliary tape, and if a blank is reached after such a pair, then add a 0 at the first blank in the auxiliary tape; then mark the current end of the auxiliary tape with a symbol  $\spadesuit$  and return from the end of the main tape, erasing it and adding a 1 to the end of the auxiliary tape for each  $\heartsuit$  in the main tape; now copy the 1s after  $\spadesuit$  from the auxiliary tape to the main tape (and remove these 1s from the auxiliary tape), and repeat. Each iteration has cost smaller than  $(U + U + \log U)c$  for some constant  $c$ , where  $U$  is the number of ones in the main tape; thus the total cost from input of size  $f(N)$  is smaller than  $3c(f(N) + f(N)/2 + f(N)/4 + \dots) \leq 6cf(N)$ .

and

$$Z_3 \equiv \forall \chi : \bigwedge_t \forall y : \bigwedge_{j \in \{1,2\}} \bigvee_{\sigma} \left( Y_{\sigma}^{t,j}(\chi, y) \wedge \bigwedge_{\sigma' \neq \sigma} \neg Y_{\sigma'}^{t,j}(\chi, y) \right).$$

As another example, we can change  $Z_4$  as follows. First, introduce auxiliary definition axioms:

$$W_1^t(\chi) \equiv \exists y : Z^{t,1}(\chi, y) \wedge (\forall z : (z \neq y) \rightarrow \neg Z^{t,1}(\chi, z)) \wedge (\forall z : \neg Z^{t,2}(\chi, z)),$$

$$W_2^t(\chi) \equiv \exists y : Z^{t,2}(\chi, y) \wedge (\forall z : (z \neq y) \rightarrow \neg Z^{t,2}(\chi, z)) \wedge (\forall z : \neg Z^{t,1}(\chi, z)),$$

and then write:

$$Z_4 \equiv \forall \chi : \bigwedge_t W_1^t(\chi) \wedge W_2^t(\chi).$$

Similar changes must be made to  $Z_7$  and  $Z_8$ :

$$Z_7 \equiv \forall \chi : \bigwedge_t \forall y : \bigwedge_{j \in \{1,2\}} \forall z : Y_{\sqcup}^{t,j}(\chi, y) \wedge \text{successor}(y, z) \rightarrow Y_{\sqcup}^{t,j}(\chi, z),$$

$$Z_8 \equiv \exists \chi : \bigvee_t X_{q_a}^t(\chi).$$

The changes to  $Z_5$  and  $Z_6$  are similar, but require more tedious repetition; we omit the complete expressions but explain the procedure. Basically,  $Z_5$  and  $Z_6$  encode the transitions of the Turing machine. So, instead of just taking the successor of a computation step  $\chi$ , we must operate in substeps: the successor of step  $\chi$  substep  $t$  is  $\chi$  substep  $t + 1$ , unless  $t = M$  (in which case we must move to the successor of  $\chi$ , substep 1). We can also capture the behavior of the Turing machine with two transition functions, one per tape, and it is necessary to encode each one of them appropriately. It is enough to have  $M$  different versions of  $Z_5$  and  $2M$  different versions of  $Z_6$ , each one of them responsible for one particular substep transition.

To finish, we must encode the initial conditions. Introduce:

$$\text{last}(\chi) \equiv \neg \exists y : \chi < y$$

and

$$\begin{aligned} Z_9 \equiv & \left( \forall \chi : \forall y : (\text{first}(\chi) \wedge \neg \text{last}(y)) \rightarrow Y_1^{1,1}(\chi, y) \right) \\ & \wedge (\forall \chi : \forall y : (\text{first}(\chi) \wedge \text{last}(y)) \rightarrow Y_{\sqcup}^{1,1}(\chi, y)) \\ & \wedge (\forall \chi : \forall y : \text{first}(\chi) \rightarrow Y_{\sqcup}^{1,2}(\chi, y)). \end{aligned}$$

Now  $\mathbb{P}(Z_8 | \bigwedge_{i=1}^9 Z_i) > 1/2$  for a domain of size  $f(N) + 1$  if and only if the number of interpretations that set an accepting state to true is larger than half the total number of interpretations encoding computation paths.  $\square$

**Theorem 15.** *Suppose relations have bounded arity.  $\text{INF}[\text{QF}]$  and  $\text{QINF}[\text{QF}]$  are PP-complete with respect to many-one reductions, and  $\text{DINF}[\text{QF}]$  requires constant computational effort. These results hold even if domain size is given in binary notation.*

*Proof.* Consider first  $\text{INF}[\text{Q}]$ . To prove membership, take a relational Bayesian network specification  $\mathbb{S}$  with relations  $X_1, \dots, X_n$ , all with arity no larger than  $k$ . Suppose we ground this specification on a domain of size  $N$ . To compute  $\mathbb{P}(\mathbf{Q} | \mathbf{E})$ , the only relevant groundings are the ones that are ancestors of each of the ground atoms in  $\mathbf{Q} \cup \mathbf{E}$ . Our strategy will be to bound the number of such relevant groundings. To do that, take a grounding  $X_i(a_1, \dots, a_{k_i})$  in  $\mathbf{Q} \cup \mathbf{E}$ , and suppose that  $X_i$  is not a root node in the parvariable graph. Each parent  $X_j$  of  $X_i$  in the parvariable graph may appear in several

different forms in the definition axiom related to  $X_i$ ; that is, we may have  $X_j(\mathfrak{x}_2, \mathfrak{x}_3), X_j(\mathfrak{x}_9, \mathfrak{x}_1), \dots$ , and each one of these combinations leads to a distinct grounding. There are in fact at most  $k_i^{k_i}$  ways to select individuals from the grounding  $X_i(a_1, \dots, a_{k_i})$  so as to form groundings of  $X_j$ . So for each parent of  $X_i$  in the parvariable graph there will be at most  $k^k$  relevant groundings. And each parent of these parents will again have at most  $k^k$  relevant groundings; hence there are at most  $(n-1)k^k$  relevant groundings that are ancestors of  $X_i(a_1, \dots, a_{k_i})$ . We can take the union of all groundings that are ancestors of groundings of  $\mathbf{Q} \cup \mathbf{E}$ , and the number of such groundings is still polynomial in the size of the input. Thus in polynomial time we can build a polynomially-large Bayesian network that is a fragment of the grounded Bayesian network. Then we can run a Bayesian network inference in this smaller network (an effort within PP); note that domain size is actually not important so it can be specified either in unary or binary notation. To prove hardness, note that  $\text{INF}[\text{Prop}(\wedge, \neg)]$  is PP-hard, and a propositional specification can be reproduced within QF.

Now consider  $\text{QINF}[\text{QF}]$ . To prove membership, note that even  $\text{INF}[\text{QF}]$  is in PP. To prove hardness, take an instance of  $\#3\text{SAT}(>)$  consisting of a sentence  $\phi$  in 3CNF, with propositions  $A_1, \dots, A_n$ , and an integer  $k$ . Consider the relational Bayesian network specification consisting of eight definition axioms:

$$\begin{aligned} \text{clause0}(\chi, y, z) &\equiv \neg\text{sat}(\chi) \vee \neg\text{sat}(y) \vee \neg\text{sat}(z), \\ \text{clause1}(\chi, y, z) &\equiv \neg\text{sat}(\chi) \vee \neg\text{sat}(y) \vee \text{sat}(z), \\ \text{clause2}(\chi, y, z) &\equiv \neg\text{sat}(\chi) \vee \text{sat}(y) \vee \neg\text{sat}(z), \\ &\vdots \quad \vdots \quad \vdots \\ \text{clause7}(\chi, y, z) &\equiv \text{sat}(\chi) \vee \text{sat}(y) \vee \text{sat}(z), \end{aligned}$$

and  $\mathbb{P}(\text{sat}(\chi) = 1) = 1/2$ . Now the query is just a set of assignments  $\mathbf{Q}$  ( $\mathbf{E}$  is empty) containing an assignment per clause. If a clause is  $\neg A_2 \vee A_3 \vee \neg A_1$ , then take the corresponding assignment  $\{\text{clause2}(a_2, a_3, a_1) = 1\}$ , and so on. The  $\#3\text{SAT}(>)$  problem is solved by deciding whether  $\mathbb{P}(\mathbf{Q}) > k/2^n$  with domain of size  $n$ ; hence the desired hardness is proved.

And  $\text{DINF}[\text{QF}]$  requires constant effort: in fact, domain size is not relevant to a fixed inference, as can be seen from the proof of inferential complexity above.  $\square$

**Theorem 16.** *Suppose the domain size is specified in unary notation. Then  $\text{INF}[\text{EL}]$  and  $\text{QINF}[\text{EL}]$  are PP-complete with respect to many-one reductions, even if the query contains only positive assignments, and  $\text{DINF}[\text{EL}]$  is in P.*

*Proof.*  $\text{INF}[\text{EL}]$  belongs to PP by Theorem 11 as EL belongs to FFFO<sup>2</sup>. Hardness is obtained from hardness of query complexity.

So, consider  $\text{QINF}[\text{EL}]$ . Membership follows from membership of  $\text{INF}[\text{EL}]$ , so we focus on hardness. Our strategy is to reduce  $\text{INF}[\text{Prop}(\vee)]$  to  $\text{QINF}[\text{EL}]$ , using most of the construction in the proof of Theorem 2. So take a sentence  $\phi$  in 3CNF with propositions  $A_1, \dots, A_n$  and  $m$  clauses, and an integer  $k$ . The goal is to decide whether  $\#(1\text{-in-}3)\phi > k$ . We can assume that no clause contains a repeated literal.

We start by adapting several steps in the proof of Theorem 2. First, associate each literal with a random variable  $X_{ij}$  (where  $X_{ij}$  stands for a *negated* literal). In the present proof we use a parvariable  $X(\chi)$ ; the idea is that  $\chi$  is the integer  $3(i-1) + j$  for some  $i \in \{1, \dots, n\}$  and  $j \in \{1, 2, 3\}$  (clearly we can obtain  $(i, j)$  from  $\chi$  and vice-versa). Then associate  $X$  with the assessment

$$\mathbb{P}(X(\chi) = 1) = \varepsilon,$$

where  $\varepsilon$  is exactly as in the proof for  $\text{INF}[\text{Prop}(\vee)]$ .

The next step in the proof of Theorem 2 is to introduce a number of definition axioms of the form  $Y_{iuv} \equiv X_{iu} \vee X_{iv}$ , together with assignments  $\{Y_{iuv} = 1\}$ . There are  $3m$  such axioms. Then additional axioms are added to guarantee that configurations are sensible. Note that we can compute in polynomial time the total number of definition axioms that are to be created. We denote this number

by  $N$ , as we will use it as the size of the domain. In any case, we can easily bound  $N$ : first, each clause produces 3 definition axioms as in Expression 6; second, to guarantee that configurations are sensible, every time a literal is identical to another literal, or identical to the negation of another literal, four definition axioms are inserted (there are  $3m$  literals, and for each one there may be 2 identical/negated literals in the other  $m - 1$  clauses). Thus we have that  $N \leq 3m + 4 \times 3m \times 2(m - 1) = 24m^2 - 21m$ . Suppose we order these definition axioms from 1 to  $N$  by some appropriate scheme.

To encode these  $N$  definition axioms, we introduce two other parvariables  $Y(\chi)$  and  $Z(\chi, y)$ , with definition axiom

$$Y(\chi) \equiv \exists y : Z(\chi, y) \wedge X(y)$$

and assessment

$$\mathbb{P}(Z(\chi, y) = 1) = \eta,$$

for some  $\eta$  to be determined later. The idea is this. We take a domain with size  $N$ , and for each  $\chi$  from 1 to  $N$ , we set  $Z(\chi, y)$  to 0 if  $X(\chi)$  does not appear in the definition axiom indexed by  $\chi$ , and we set  $Z(\chi, y)$  to 1 if  $X(\chi)$  appears in the definition axiom indexed by  $\chi$ . We collect all these assignments in a set  $\mathbf{E}$ . Note that  $\mathbf{E}$  in effect “creates” all the desired definition axioms by selecting two instances of  $X$  per instance of  $Y$ .

Note that if we enforce  $\{Y(\chi) = 1\}$  for all  $\chi$ , we obtain the same construction used in the proof of Theorem 2, with one difference: in that proof we had  $3m$  variables  $X_{ij}$ , while here we have  $N$  variables  $X(\chi)$  (note that  $N \geq 3m$ , and  $N > 3m$  for  $m > 1$ ).

Consider grounding this relational Bayesian network specification and computing

$$\mathbb{P}(X(1) = x_1, \dots, X(N) = x_N, Y(1) = y_1, \dots, Y(N) = y_N | \mathbf{E}).$$

This distribution is encoded by a Bayesian network consisting of nodes  $X(1), \dots, X(N)$  and nodes  $Y(1), \dots, Y(N)$ , where all nodes  $Z(\chi, y)$  are removed as they are set by  $\mathbf{E}$ ; also, each node  $Y(\chi)$  has two parents, and all nodes  $X(3m + 1), \dots, X(N)$  have no children. Denote by  $\mathbf{L}$  a generic configuration of  $X(1), \dots, X(3m)$ , and by  $\mathbf{Q}$  a configuration of  $Y(1), \dots, Y(N)$  where all variables are assigned value 1. As in the proof of Theorem 2, we have  $\mathbb{P}(\mathbf{L}) = \alpha$  if  $\mathbf{L}$  is gratifying-sensible-respectful, and  $\mathbb{P}(\mathbf{L}) \leq \beta$  if  $\mathbf{L}$  is respectful but not gratifying. If  $\#(1\text{-in-}3)\phi > k$ , then  $\mathbb{P}(\mathbf{Q} | \mathbf{E}) = \sum_{\mathbf{L}} \mathbb{P}(\mathbf{L}, \mathbf{Q}) \geq (k + 1)\alpha$ . And if  $\#(1\text{-in-}3)\phi \leq k$ , then  $\mathbb{P}(\mathbf{Q} | \mathbf{E}) \leq k\alpha + 4^m\beta$ . Define  $\delta_1 = (k + 1)\alpha$  and  $\delta_2 = k\alpha + 4^m\beta$  and choose  $\varepsilon < 1/(1 + 4^m)$  to guarantee that  $\delta_1 > \delta_2$ , so that we can differentiate between the two cases with an inference.

We have thus solved our original problem using a fixed Bayesian network specification plus a query  $(\mathbf{Q}, \mathbf{E})$ . Hence PP-hardness of QINF[EL] obtains. However, note that  $\mathbf{Q}$  contains only positive assignments, but  $\mathbf{E}$  contains both positive and negative assignments. We now constrain ourselves to positive assignments.

Denote by  $\mathbf{E}_1$  the assignments of the form  $\{Z(\chi, y) = 1\}$  in  $\mathbf{E}$ , and denote by  $\mathbf{E}_0$  the assignments of the form  $\{Z(\chi, y) = 0\}$  in  $\mathbf{E}$ . Consider:

$$\mathbb{P}(\mathbf{Q} | \mathbf{E}_1) = \mathbb{P}(\mathbf{Q} | \mathbf{E}_0, \mathbf{E}_1) \mathbb{P}(\mathbf{E}_0 | \mathbf{E}_1) + \mathbb{P}(\mathbf{Q} | \mathbf{E}_0^c, \mathbf{E}_1) \mathbb{P}(\mathbf{E}_0^c | \mathbf{E}_1),$$

where  $\mathbf{E}_0^c$  is the event consisting of configurations of those variables that appear in  $\mathbf{E}_0$  such that at least one of these variables is assigned 1 (of course, such variables are assigned 0 in  $\mathbf{E}_0$ ).

We have that  $\mathbb{P}(\mathbf{Q} | \mathbf{E}_0, \mathbf{E}_1) = \mathbb{P}(\mathbf{Q} | \mathbf{E})$  by definition. And variables in  $\mathbf{E}_0$  and  $\mathbf{E}_1$  are independent, hence  $\mathbb{P}(\mathbf{E}_0 | \mathbf{E}_1) = \mathbb{P}(\mathbf{E}_0) = (1 - \eta)^M$  where  $M$  is the number of variables in  $\mathbf{E}_0$  (so  $M \leq N^2$ ). Consequently,  $\mathbb{P}(\mathbf{E}_0^c | \mathbf{E}_1) = 1 - (1 - \eta)^M$ . Thus we obtain:

$$\mathbb{P}(\mathbf{Q} | \mathbf{E}_1) = (1 - \eta)^M \mathbb{P}(\mathbf{Q} | \mathbf{E}) + (1 - (1 - \eta)^M) \mathbb{P}(\mathbf{Q} | \mathbf{E}_0^c, \mathbf{E}_1).$$

Now reason as follows. If  $\#(1\text{-in-}3)\phi > k$ , then  $\mathbb{P}(\mathbf{Q} | \mathbf{E}_1) \geq (1 - \eta)^M \delta_1$ . And if  $\#(1\text{-in-}3)\phi \leq k$ , then  $\mathbb{P}(\mathbf{Q} | \mathbf{E}_1) \leq (1 - (1 - \eta)^M) + (1 - \eta)^M \delta_2$ . To guarantee that  $(1 - \eta)^M \delta_1 > (1 - (1 - \eta)^M) + (1 - \eta)^M \delta_2$ , we must have  $(1 - \eta)^M > 1/(1 + \delta_1 - \delta_2)$ . We do so by selecting  $\eta$  appropriately. Note first that



$1/(1 + \delta_1 - \delta_2) \in (0, 1)$  by our choice of  $\varepsilon$ ; note also that  $1 + (x - 1)/M > x^{1/M}$  for any  $x \in (0, 1)$ , so select

$$1 - \eta > 1 + \left( \frac{1}{1 + \delta_1 - \delta_2} - 1 \right) / M;$$

that is,  $\eta < (1 - 1/(1 + \delta_1 - \delta_2))/M$ . By doing so, we can differentiate between the two cases with an inference, so the desired hardness is proved.

Domain complexity is polynomial because EL is in FFFO<sup>2</sup> [133, 135].  $\square$

**Theorem 17.** *Suppose the domain size is specified in unary notation. Then DINF[DLLite<sup>nf</sup>] is in P; also, INF[DLLite<sup>nf</sup>] and QINF[DLLite<sup>nf</sup>] are in P when the query  $(\mathbf{Q}, \mathbf{E})$  contains only positive assignments.*

*Proof.* We prove the polynomial complexity of INF[DLLite] with positive queries by a quadratic-time reduction to multiple problems of counting weighted edge covers with uniform weights in a particular class of graphs. Then we use the fact that the latter problem can be solved in quadratic time (hence the total time is quadratic).

From now on we simply use  $\mathbf{Q}$  to refer to a set of assignments whose probability is of interest.

We first transform the relational Bayesian network specification into an equal-probability model. Collapse each role  $r$  and its inverse  $r^-$  into a single node  $r$ . For each (collapsed) role  $r$ , insert variables  $\mathbf{e}_r \equiv \exists r$  and  $\mathbf{e}_r^- \equiv \exists r^-$ ; replace each appearance of the formula  $\exists r$  by the variable  $\mathbf{e}_r$ , and each appearance of  $\exists r^-$  by  $\mathbf{e}_r^-$ . This transformation does not change the probability of  $\mathbf{Q}$ , and it allows us to easily refer to groundings of formulas  $\exists r$  and  $\exists r^-$  as groundings of  $\mathbf{e}_r$  and  $\mathbf{e}_r^-$ , respectively.

Observe that only the nodes with assignments in  $\mathbf{Q}$  and their ancestors are relevant for the computation of  $\mathbb{P}(\mathbf{Q})$ , as every other node in the Bayesian network is barren [36]. Hence, we can assume without loss of generality that  $\mathbf{Q}$  contains only leaves of the network. If  $\mathbf{Q}$  contains only root nodes, then  $\mathbb{P}(\mathbf{Q})$  can be computed trivially as the product of marginal probabilities which are readily available from the specification. Thus assume that  $\mathbf{Q}$  assigns a positive value to at least one leaf grounding  $\mathbf{s}(a)$ , where  $a$  is some individual in the domain. Then by construction  $\mathbf{s}(a)$  is associated with a logical sentence  $X_1 \wedge \dots \wedge X_k$ , where each  $X_i$  is either a grounding of non-primitive unary relation in individual  $a$ , a grounding of a primitive unary relation in  $a$ , or the negation of a grounding of a primitive unary relation in  $a$ . It follows that  $\mathbb{P}(\mathbf{Q}) = \mathbb{P}(\mathbf{s}(a) = 1 | X_1 = 1, \dots, X_k = 1) \mathbb{P}(\mathbf{Q}') = \mathbb{P}(\mathbf{Q}')$ , where  $\mathbf{Q}'$  is  $\mathbf{Q}$  after removing the assignment  $\mathbf{s}(a) = 1$  and adding the assignments  $\{X_1 = 1, \dots, X_k = 1\}$ . Now it might be that  $\mathbf{Q}'$  contains both the assignments  $\{X_i = 1\}$  and  $\{X_i = 0\}$ . Then  $\mathbb{P}(\mathbf{Q}) = 0$  (this can be verified efficiently). So assume there are no such inconsistencies. The problem of computing  $\mathbb{P}(\mathbf{Q})$  boils down to computing  $\mathbb{P}(\mathbf{Q}')$ ; in the latter problem the node  $\mathbf{s}(a)$  is discarded for being barren. Moreover, we can replace any assignment  $\{\neg r(a) = 1\}$  in  $\mathbf{Q}'$  for some primitive concept  $r$  with the equivalent assignment  $\{r(a) = 0\}$ . By repeating this procedure for all internal nodes which are not groundings of  $\mathbf{e}_r$  or  $\mathbf{e}_r^-$ , we end up with a set  $\mathbf{A}$  containing positive assignments of groundings of roles and of concepts  $\mathbf{e}_r$  and  $\mathbf{e}_r^-$ , and (not necessarily positive) assignments of groundings of primitive concepts. Each grounding of a primitive concept or role is (a root node hence) marginally independent from all other groundings in  $\mathbf{A}$ ; hence  $\mathbb{P}(\mathbf{A}) = \mathbb{P}(\mathbf{B} | \mathbf{C}) \prod_i \mathbb{P}(A_i)$ , where each  $A_i$  is an assignment to a root node,  $\mathbf{B}$  are (positive) assignments to groundings of concepts  $\mathbf{e}_r$  and  $\mathbf{e}_r^-$  for relations  $r$ , and  $\mathbf{C} \subseteq \{A_1, A_2, \dots\}$  are groundings of roles (if  $\mathbf{C}$  is empty then assume it expresses a tautology). Since the marginal probabilities  $\mathbb{P}(A_i)$  are available from the specification the joint  $\prod_i \mathbb{P}(A_i)$  can be computed in linear time in the input. We thus focus on computing  $\mathbb{P}(\mathbf{B} | \mathbf{C})$  as defined (if  $\mathbf{B}$  is empty, we are done). To recap,  $\mathbf{B}$  is a set of assignments  $\mathbf{e}_r(a) = 1$  and  $\mathbf{e}_r^-(b) = 1$  and  $\mathbf{C}$  is a set of assignments  $r(c, d) = 1$  for arbitrary roles  $r$  and individuals  $a, b, c$  and  $d$ .

For a role  $r$ , let  $\mathcal{D}_r$  be the set of individuals  $a \in \mathcal{D}$  such that  $\mathbf{e}_r(a) = 1$  is in  $\mathbf{B}$ , and let  $\mathcal{D}_r^-$  be the set of individuals  $a \in \mathcal{D}$  such that  $\mathbf{B}$  contains  $\mathbf{e}_r^-(a) = 1$ . Let  $\text{gr}(r)$  be the set of all groundings of relation  $r$ , and let  $r_1, \dots, r_k$  be the roles in the (relational) network. By the factorization property of Bayesian networks it follows that

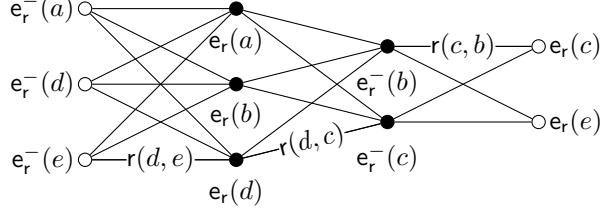


Figure 15: Representing assignments by graphs.

$$\mathbb{P}(\mathbf{B}|\mathbf{C}) = \sum_{\text{gr}(\mathbf{r}_1)} \cdots \sum_{\text{gr}(\mathbf{r}_k)} \prod_{i=1}^k \prod_{a \in \mathcal{D}_{r_i}} \mathbb{P}(\mathbf{e}_{r_i}(a) = 1 | \text{pa}(\mathbf{e}_{r_i}(a)), \mathbf{C}) \times \prod_{a \in \mathcal{D}_{r_i}^-} \mathbb{P}(\mathbf{e}_{r_i}^-(a) = 1 | \text{pa}(\mathbf{e}_{r_i}^-(a)), \mathbf{C}) \mathbb{P}(\text{gr}(\mathbf{r}_k) | \mathbf{C}),$$

which by distributing the products over sums is equal to

$$\prod_{i=1}^k \sum_{\text{gr}(\mathbf{r}_i)} \prod_{a \in \mathcal{D}_r} \mathbb{P}(\mathbf{e}_r(a) = 1 | \text{pa}(\mathbf{e}_r(a)), \mathbf{C}) \times \prod_{a \in \mathcal{D}_r^-} \mathbb{P}(\mathbf{e}_r^-(a) = 1 | \text{pa}(\mathbf{e}_r^-(a)), \mathbf{C}) \mathbb{P}(\text{gr}(\mathbf{r}_k) | \mathbf{C}).$$

Consider an assignment  $\mathbf{r}(a, b) = 1$  in  $\mathbf{C}$ . By construction, the children of the grounding  $\mathbf{r}(a, b)$  are  $\mathbf{e}_r(a)$  and  $\mathbf{e}_r^-(b)$ . Moreover, the assignment  $\mathbf{r}(a, b) = 1$  implies that  $\mathbb{P}(\mathbf{e}_r(a) = 1 | \text{pa}(\mathbf{e}_r(a)), \mathbf{C}) = 1$  (for any assignment to the other parents) and  $\mathbb{P}(\mathbf{e}_r^-(b) = 1 | \text{pa}(\mathbf{e}_r^-(b)), \mathbf{C}) = 1$  (for any assignment to the other parents). This is equivalent in the factorization above to removing  $\mathbf{r}(a, b)$  from  $\mathbf{C}$  (as it is independent of all other groundings), and removing individuals  $a$  from  $\mathcal{D}_r$  and  $b$  from  $\mathcal{D}_r^-$ . So repeat this procedure for every grounding in  $\mathbf{C}$  until this set is empty (this can be done in polynomial time). The inference problem becomes one of computing

$$\gamma(\mathbf{r}) = \sum_{\text{gr}(\mathbf{r}_i)} \prod_{a \in \mathcal{D}_r} \mathbb{P}(\mathbf{e}_r(a) = 1 | \text{pa}(\mathbf{e}_r(a))) \prod_{a \in \mathcal{D}_r^-} \mathbb{P}(\mathbf{e}_r^-(a) = 1 | \text{pa}(\mathbf{e}_r^-(a))) \mathbb{P}(\text{gr}(\mathbf{r}_k))$$

for every relation  $r_i$ ,  $i = 1, \dots, k$ . We will show that this problem can be reduced to a tractable instance of counting weighted edge covers.

To this end, consider the graph  $G$  whose node set  $V$  can be partitioned into sets  $V_1 = \{\mathbf{e}_r^-(a) : a \in \mathcal{D} \setminus \mathcal{D}_r^-\}$ ,  $V_2 = \{\mathbf{e}_r(a) : a \in \mathcal{D}_r\}$ ,  $V_3 = \{\mathbf{e}_r^-(a) : a \in \mathcal{D}_r^-\}$ ,  $V_4 = \{\mathbf{e}_r(a) : a \in \mathcal{D} \setminus \mathcal{D}_r\}$ , and for  $i = 1, 2, 3$  the graph obtained by considering nodes  $V_i \cup V_{i+1}$  is bipartite complete. An edge with endpoints  $\mathbf{e}_r(a)$  and  $\mathbf{e}_r^-(b)$  represents the grounding  $\mathbf{r}(a, b)$ ; we identify every edge with its corresponding grounding. We call this graph the *intersection graph* of  $\mathbf{B}$  with respect to  $\mathbf{r}$  and  $\mathcal{D}$ . The parents of a node in the graph correspond exactly to the parents of the node in the Bayesian network. For example, the graph in Figure 15 represents the assignments  $\mathbf{B} = \{\mathbf{e}_r(a) = 1, \mathbf{e}_r(b) = 1, \mathbf{e}_r(d) = 1, \mathbf{e}_r^-(b) = 1, \mathbf{e}_r^-(c) = 1\}$ , with respect to domain  $\mathcal{D} = \{a, b, c, d, e\}$ . The black nodes (resp., white nodes) represent groundings in (resp., not in)  $\mathbf{B}$ . For clarity's sake, we label only a few edges.

Before showing the equivalence between the inference problem and counting edges covers, we need to introduce some graph-theoretic notions and notation. Consider a (simple, undirected) graph  $G = (V, E)$ . Denote by  $E_G(u)$  the set of edges incident on a node  $u \in V$ , and by  $N_G(u)$  the open neighborhood of  $u$ . For  $U \subseteq V$ , we say that  $C \subseteq E$  is a  $U$ -cover if for each node  $u \in U$  there is an edge  $e \in C$  incident in  $u$  (i.e.,  $e \in E_G(u)$ ). For any fixed real  $\lambda$ , we say that  $\lambda^{|C|}$  is the weight of cover  $C$ . The *partition function* of  $G$  is  $Z(G, U, \lambda) = \sum_{C \in EC(G, U)} \lambda^{|C|}$ , where  $U \subseteq V$ ,  $EC(G, U)$  is the set

of  $U$ -covers of  $G$  and  $\lambda$  is a positive real. If  $\lambda = 1$  and  $U = V$ , the partition function is the number of edge covers. The following result connects counting edge covers to marginal inference in DL-Lite Bayesian networks.

**Lemma 1.** *Let  $G = (V_1, V_2, V_3, V_4, E)$  be the intersection graph of  $\mathbf{B}$  with respect to a relation  $r$  and domain  $\mathcal{D}$ . Then  $\gamma(r) = Z(G, V_2 \cup V_3, \alpha/(1 - \alpha))/(1 - \alpha)^{|E|}$ , where  $\alpha = \mathbb{P}(r(\chi, y))$ .*

*Proof of Lemma 1.* Let  $B = V_2 \cup V_3$ , and consider a  $B$ -cover  $C$ . The assignment that sets to true all groundings  $r(a, b)$  corresponding to edges in  $C$ , and sets to false the remaining groundings of  $r$  makes  $\mathbb{P}(\mathbf{e}_r(a) = 1 | \text{pa}(\mathbf{e}_r(a))) = \mathbb{P}(\mathbf{e}_r^-(b) = 1 | \text{pa}(\mathbf{e}_r^-(b))) = 1$  for every  $a \in \mathcal{D}_r$  and  $b \in \mathcal{D}_r^-$ ; it makes  $\mathbb{P}(\mathbf{gr}(r)) = \mathbb{P}(r)^{|C|} (1 - \mathbb{P}(r))^{|E| - |C|} = (1 - \alpha)^{|E|} \alpha^{|C|} / (1 - \alpha)^{|C|}$ , which is the weight of the cover  $C$  scaled by  $(1 - \alpha)^{|E|}$ . Now consider a set of edges  $C$  which is not a  $B$ -cover and obtains an assignment to groundings  $\mathbf{gr}(r)$  as before. There is at least one node in  $B$  that does not contain any incident edges in  $C$ . Assume that node is  $\mathbf{e}(a)$ ; then all parents of  $\mathbf{e}(a)$  are assigned false, which implies that  $\mathbb{P}(\mathbf{e}_r(a) = 1 | \text{pa}(\mathbf{e}_r(a))) = 0$ . The same is true if the node not covered is a grounding  $\mathbf{e}^-(a)$ . Hence, for each  $B$ -cover  $C$  the probability of the corresponding assignment equals its weight up to the factor  $(1 - \alpha)^{|E|}$ . And for each edge set  $C$  which is not a  $B$ -cover its corresponding assignment has probability zero.  $\square$

We have thus established that, if a particular class of edge cover counting problems is polynomial, then marginal inference in DL-Lite Bayesian networks is also polynomial. Because the former is shown to be true in  $\mathbf{B}$ , this concludes the proof of Theorem 17.  $\square$

**Theorem 18.** *Given a relational Bayesian network  $\mathbb{S}$  based on  $\text{DLLite}^{\text{nf}}$ , a set of positive assignments to grounded relations  $\mathbf{E}$ , and a domain size  $N$  in unary notation,  $\text{MLE}(\mathbb{S}, \mathbf{E}, N)$  can be solved in polynomial time.*

*Proof.* In this theorem we are interested in finding an assignment  $\mathbf{X}$  to all groundings that maximizes  $\mathbb{P}(\mathbf{X} \wedge \mathbf{E})$ , where  $\mathbf{E}$  is a set of positive assignments. Perform the substitution of formulas  $\exists r$  and  $\exists r^-$  by logically equivalent concepts  $\mathbf{e}_r$  and  $\mathbf{e}_r^-$  as before. Consider a non-root grounding  $\mathbf{s}(a)$  in  $\mathbf{E}$  which is not the grounding of  $\mathbf{e}_r$  or  $\mathbf{e}_r^-$ ; by construction,  $\mathbf{s}(a)$  is logically equivalent to a conjunction  $X_1 \wedge \dots \wedge X_k$ , where  $X_1, \dots, X_k$  are unary groundings. Because  $\mathbf{s}(a)$  is assigned to true, any assignment  $\mathbf{X}$  with nonzero probability assigns  $X_1, \dots, X_k$  to true. Moreover, since  $\mathbf{s}(a)$  is an internal node, its corresponding probability is one. Hence, if we include all the assignments  $X_i = 1$  to its parents in  $\mathbf{E}$ , the MPE value does not change. As in the computation of inference, we might generate an inconsistency when setting the values of parents; in this case halt and return zero (and an arbitrary assignment). So assume we repeated this procedure until  $\mathbf{E}$  contains all ancestors of the original groundings which are groundings of unary relations, and that no inconsistency was found. Note that at this point we only need to assign values to nodes which are either not ancestors of any node in the original set  $\mathbf{E}$ , and to groundings of (collapsed) roles  $r$ .

Consider the groundings of primitive concepts  $r$  which are not ancestors of any grounding in  $\mathbf{E}$ . Setting its value to maximize its marginal probability does not introduce any inconsistency with respect to  $\mathbf{E}$ . Moreover, for any assignment to these groundings, we can find a consistent assignment to the remaining groundings (which are internal nodes and not ancestors of  $\mathbf{E}$ ), that is, an assignment which assigns positive probability. Since this is the maximum probability we can obtain for these groundings, this is a partial optimum assignment.

We are thus only left with the problem of assigning values to the groundings of relations  $r$  which are ancestors of  $\mathbf{E}$ . Consider a relation  $r$  such that  $\mathbb{P}(r) \geq 1/2$ . Then assigning all groundings of  $r$  to true maximizes their marginal probability and satisfies the logical equivalences of all groundings in  $\mathbf{E}$ . Hence, this is a maximum assignment (and its value can be computed efficiently). So assume there is a relation  $r$  with  $\mathbb{P}(r) < 1/2$  such that a grounding of  $\mathbf{e}_r$  or  $\mathbf{e}_r^-$  appear in  $\mathbf{E}$ . In this case, the greedy assignment sets every grounding of  $r$ ; however, such an assignment is inconsistent with the logical equivalence of  $\mathbf{e}_r$  and  $\mathbf{e}_r^-$ , hence obtains probability zero. Now consider an assignment that assigns exactly one grounding  $r(a, b)$  to true and all the other to false. This assignment is consistent

with  $e_r(a)$  and  $e_r(b)$ , and maximizes the probability; any assignment that sets more groundings to true has a lower probability since it replaces a term  $1 - \mathbb{P}(r) \geq 1/2$  with a term  $\mathbb{P}(r) < 1/2$  in the joint probability. More generally, to maximize the joint probability we need to assign to true as few groundings  $r(a, b)$  which are ancestors of  $\mathbf{E}$  as possible. This is equivalent to a minimum cardinality edge covering problem as follows.

For every relation  $r$  in the relational network, construct the bipartite complete graph  $G_r = (V_1, V_2, E)$  such that  $V_1$  is the set of groundings  $e_r(a)$  that appears and have no parent  $r(a, b)$  in  $\mathbf{E}$ , and  $V_2$  is the set of groundings  $e_r^-(b)$  that appears and have no parents in  $\mathbf{E}$ . We identify an edge connecting  $e_r(a)$  and  $e_r^-(b)$  with the grounding  $r(a, b)$ . For any set  $C \subseteq E$ , construct an assignment by attaching true to the groundings  $r(a, b)$  in  $C$  and false to every other grounding  $r(a, b)$ . This assignment is consistent with  $\mathbf{E}$  if and only if  $C$  is an edge cover; hence the minimum cardinality edge cover maximizes the joint probability (it is consistent with  $\mathbf{E}$  and attaches true to the least number of groundings of  $\mathbf{r}$ ). This concludes the proof of Theorem 18.  $\square$

**Theorem 19.** *INF[PLATE] and QINF[PLATE] are PP-complete with respect to many-one reductions, and DINF[PLATE] requires constant computational effort. These results hold even if the domain size is given in binary notation.*

*Proof.* Consider first INF[PLATES]. To prove membership, take a plate model with relations  $X_1, \dots, X_n$ . Suppose we ground this specification on a domain of size  $N$ . To compute  $\mathbb{P}(\mathbf{Q}|\mathbf{E})$ , the only relevant groundings are the ones that are ancestors of each of the ground atoms in  $\mathbf{Q} \cup \mathbf{E}$ . Our strategy will be to bound the number of such relevant groundings. To do that, take a grounding  $X_i(a_1, \dots, a_{k_i})$  in  $\mathbf{Q} \cup \mathbf{E}$ , and suppose that  $X_i$  is not a root node. Each parent  $X_j$  of  $X_i$  may appear once in the definition axiom related to  $X_i$ . And each parent of these parents will again have a limited number of parent groundings; in the end there are at most  $(n - 1)$  relevant groundings that are ancestors of  $X_i(a_1, \dots, a_{k_i})$ . We can take the union of all groundings that are ancestors of groundings of  $\mathbf{Q} \cup \mathbf{E}$ , and the number of such groundings is still polynomial in the size of the input. Thus in polynomial time we can build a polynomially-large Bayesian network that is a fragment of the grounded Bayesian network. Then we can run a Bayesian network inference in this smaller network (an effort within PP); note that domain size is actually not important so it can be specified either in unary or binary notation. To prove hardness, note that INF[Prop( $\wedge, \neg$ )] is PP-hard, and a propositional specification can be reproduced within PLATES.

Now consider QINF[PLATES]. First, to prove membership, note that even INF[PLATES] is in PP. To prove hardness, reproduce the proof of Theorem 15 by encoding a #3SAT( $>$ ) problem, specified by sentence  $\phi$  and integer  $k$ , with the definition axioms:

$$\begin{aligned} \text{clause0}(\chi, y, z) &\equiv \neg\text{left}(\chi) \vee \neg\text{middle}(y) \vee \neg\text{right}(z), \\ \text{clause1}(\chi, y, z) &\equiv \neg\text{left}(\chi) \vee \neg\text{middle}(y) \vee \text{right}(z), \\ \text{clause2}(\chi, y, z) &\equiv \neg\text{left}(\chi) \vee \text{middle}(y) \vee \neg\text{right}(z), \\ &\vdots \quad \vdots \quad \vdots \\ \text{clause7}(\chi, y, z) &\equiv \text{left}(\chi) \vee \text{middle}(y) \vee \text{right}(z), \\ \text{equal}(\chi, y, z) &\equiv \text{left}(\chi) \leftrightarrow \text{middle}(y) \leftrightarrow \text{right}(z), \end{aligned}$$

and  $\mathbb{P}(\text{left}(\chi) = 1) = \mathbb{P}(\text{middle}(\chi) = 1) = \mathbb{P}(\text{right}(\chi) = 1) 1/2$ . The resulting plate model is depicted in Figure 16. The query is again just a set of assignments  $\mathbf{Q}$  ( $\mathbf{E}$  is empty) containing an assignment per clause. If a clause is  $\neg A_2 \vee A_3 \vee \neg A_1$ , then take the corresponding assignment  $\{\text{clause2}(a_2, a_3, a_1) = 1\}$ , and so on. Moreover, add the assignments  $\{\text{equal}(a_i, a_i, a_i) = 1\}$  for each  $i \in \{1, \dots, n\}$ , to guarantee that left, middle and right have identical truth assignments for all elements of the domain. The #3SAT( $>$ ) is solved by deciding whether  $\mathbb{P}(\mathbf{Q}) > k/2^n$  with domain of size  $n$ ; hence the desired hardness is proved.

And DINF[PLATES] requires constant effort: in fact, domain size is not relevant to a fixed inference, as can be seen from the proof of inferential complexity above.  $\square$

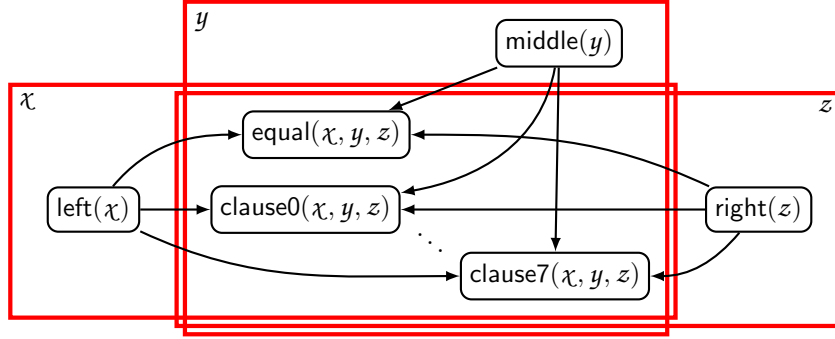


Figure 16: A plate model that decides a  $\#3\text{SAT}(>)$  problem.

**Theorem 20.** Consider the class of functions that gets as input a relational Bayesian network specification based on FFFO, a domain size  $N$  (in binary or unary notation), and a set of assignments  $\mathbf{Q}$ , and returns  $\mathbb{P}(\mathbf{Q})$ . This class of functions is  $\#EXP$ -equivalent.

*Proof.* Build a relational Bayesian network specification as in the proof of Theorem 4. Note that the  $p = \mathbb{P}(\mathbf{E} \wedge \bigwedge_{i=1}^6 Z_i)$  is the probability that a tiling is built satisfying all horizontal and vertical restrictions and the initial condition, and moreover containing the accepting state  $q_a$ .

If we can recover the number of tilings of the torus from this probability, we obtain the number of accepting computations of the exponentially-bounded Turing machine we started with. Assume we have  $p$ . There are  $2^{2n}$  elements in our domain; if the plate model is grounded, there are  $2^{2n}(2n+c)$  grounded root random variables, hence there are  $2^{2^{2n}(2n+c)}$  interpretations. Hence  $p \times 2^{2^{2n}(2n+c)}$  is the number of truth assignments that build the board satisfying all horizontal and vertical constraints and the initial conditions. However, this number is *not* equal to the number of tilings of the board. To see this, consider the grounded Bayesian network where each  $a$  in the domain is associated with a “slice” containing groundings  $X_i(a)$ ,  $Y_i(a)$ ,  $C_j(a)$  and so on. If a particular configuration of these indicator variables corresponds to a tiling, then we can produce the same tiling by permuting all elements of the domain with respect to the slices of the network. Intuitively, we can fix a tiling and imagine that we are labelling each point of the torus with an element of the domain; clearly every permutation of these labels produces the same tiling (this intuition is appropriate because each  $a$  corresponds to a different point in the torus). So, in order to produce the number of tilings of the torus, we must compute  $p \times 2^{2^{2n}(2n+c)} / (2^{2n}!)$ , where we divide the number of satisfying truth assignments by the number of repeated tilings.  $\square$

**Theorem 21.** Consider the class of functions that gets as input a relational Bayesian network specification based on FFFO with relations with bounded arity, a domain size  $N$  in unary notation, and a set of assignments  $\mathbf{Q}$ , and returns  $\mathbb{P}(\mathbf{Q})$ . This class of functions is  $\sharp PSPACE$ -equivalent.

*Proof.* First we describe a counting Turing machine that produces a count proportional to  $\mathbb{P}(\mathbf{Q})$  using a polynomial number of nondeterministic guesses. This nondeterministic machine guesses a truth assignment for each one of the polynomially-many grounded root nodes (and writes the guess in the working tape). Note that each grounded root node  $X$  is associated with an assessment  $\mathbb{P}(X = 1) = c/d$ , where  $c$  and  $d$  are integers. The machine must replicate its computation paths to handle such rational assessments exactly as in the proof of Theorem 7. The machine then verifies, in each computation path, whether the guessed truth assignment satisfies  $\mathbf{Q}$ ; if it does, then accept; if not, then reject. Denote by  $R$  the number of grounded root nodes and by  $\#A$  the number of accepting paths of this machine; then  $\mathbb{P}(\mathbf{Q}) = \#A/2^R$ .

Now we show that  $\mathbf{Q}$  is  $\sharp PSPACE$ -hard with respect to weighted reductions. Define  $\varphi(x_1, \dots, x_m)$

to be a quantified Boolean formula with free logvars  $\chi_1, \dots, \chi_m$ :

$$\forall y_1 : Q_2 y_2 : \dots Q_M \chi_M : \phi(\chi_1, \dots, \chi_m),$$

where each logvar can only be true or false, each  $Q_j$  is a quantifier (either  $\forall$  or  $\exists$ ). And define  $\#\varphi$  to be the number of instances of  $\chi_1, \dots, \chi_m$  such that  $\varphi(\chi_1, \dots, \chi_m)$  is true. Denote by  $\sharp\text{QBF}$  the function that gets a formula  $\varphi(\chi_1, \dots, \chi_m)$  and returns  $\#\varphi$ ; Ladner shows that  $\sharp\text{QBF}$  is  $\sharp\text{PSPACE}$ -complete [77, Theorem 5(2)]. So, adapt the hardness proof of Theorem 10: introduce the definition axiom

$$Y \equiv \forall y_1 : \dots Q_m y_m : \phi'(X_1, \dots, X_m),$$

where  $\phi'$  has the same structure of  $\phi$  but logvars are replaced as follows. First, each  $\chi_j$  is replaced by a relation  $X_j$  of arity zero (that is, a proposition). Second, each logvar  $y_j$  is replaced by the atom  $X(y_j)$  where  $X$  is a fresh unary relation. These relations are associated with assessments  $\mathbb{P}(X_j = 1) = 1/2$  and  $\mathbb{P}(X(\chi) = 1) = 1/2$ . This completes the relational Bayesian network specification. Now for domain  $\{0, 1\}$ , first compute  $\mathbb{P}(\mathbf{Q})$  for  $\mathbf{Q} = \{Y = 1, X(0) = 0, X(1) = 1\}$  and then compute  $2^m(\mathbb{P}(\mathbf{Q})/(1/4))$ . The latter number is the desired value of  $\sharp\text{QBF}$ ; note that  $\mathbb{P}(\mathbf{Q})/(1/4) = \mathbb{P}(Y = 1 | X(0) = 0, X(1) = 1)$ .  $\square$

**Theorem 22.** *Consider the class of functions that gets as input a relational Bayesian network specification based on  $\text{FFFO}^k$  for  $k \geq 2$ , a domain size  $N$  in unary notation, and a set of assignments  $\mathbf{Q}$ , and returns  $\mathbb{P}(\mathbf{Q})$ . This class of functions is  $\sharp\text{P}$ -equivalent.*

*Proof.* Hardness is trivial: even  $\text{Prop}(\wedge, \neg)$  is  $\sharp\text{P}$ -equivalent, as  $\text{Prop}(\wedge, \neg)$  suffice to specify any propositional Bayesian network, and equivalence then obtains [115]. To prove membership, use the Turing machine described in the proof of membership in Theorem 11 without assignments  $\mathbf{E}$  (that is, the machine only processes  $\mathbf{Q}$ ) and without Park's construction. At the end the machine produces the number  $\#A$  of computation paths that satisfy  $\mathbf{Q}$ ; then return  $\#A/2^R$ , where  $R$  is the number of grounded root nodes.  $\square$

**Theorem 23.** *Consider the class of functions that get as input a plate model based on  $\text{FFFO}$ , a domain size  $N$  in unary notation, and a set of assignments  $\mathbf{Q}$ , and returns  $\mathbb{P}(\mathbf{Q})$ . This class of functions is  $\sharp\text{P}$ -equivalent.*

*Proof.* Hardness is trivial: a propositional Bayesian network can be encoded with a plate model. To prove membership, build the same fragment of the grounded Bayesian network as described in the proof of Theorem 19: inference with the plate model is then reduced to inference with this polynomially large Bayesian network.  $\square$

## B A tractable class of model counting problems

“Model counting” usually refers to the problem of counting the number of satisfying truth-value assignments of a given Boolean formula. Many problems in artificial intelligence and combinatorial optimization can be either specialized to or generalized from model counting. For instance, propositional satisfiability (i.e., the problem of deciding whether a satisfying truth-value assignment exists) is a special case of model counting; probabilistic reasoning in graphical models such as Bayesian networks can be reduced to a weighted variant of model counting [6, 36]; validity of conformal plans can be formulated as model counting [96]. Thus, characterizing the theoretical complexity of the problem is both of practical and theoretical interest.

In unrestricted form, the problem is complete for the class  $\sharp\text{P}$  (with respect to various reductions). Even very restrictive versions of the problem are complete for  $\sharp\text{P}$ . For example, the problem is  $\sharp\text{P}$ -complete even when the formulas are in conjunctive normal form with two variables per clause, there is no negation, and the variables can be partitioned into two sets such that no clause contains two variables in the same block [110]. The problem is also  $\sharp\text{P}$ -complete when the formula is monotone and each

variable appears at most twice, or when the formula is monotone, the clauses contain two variables and each variable appears at most  $k$  times for any  $k \geq 5$  [129]. A few tractable classes have been found: for example, Roth [115] developed an algorithm for counting the number of satisfying assignments of formulas in conjunctive normal form with two variables per clause, each variable appearing in at most two clauses. Relaxing the constraint on the number of variables per clause takes us back to intractability: model counting restricted to formulas in conjunctive normal form with variables appearing in at most two clauses is  $\#P$ -complete [14].

Researchers have also investigated the complexity with respect to the graphical representation of formulas. Computing the number of satisfying assignments for monotone formulas in conjunctive normal form, with at most two variables per clause, with each variable appearing at most four times is  $\#P$ -complete even when the primal graph (where nodes are variables and an edge connects variables that coappear in a clause) is bipartite and planar [129]. The problem is also  $\#P$ -complete for monotone conjunctive normal form formulas whose primal graph is 3-regular, bipartite and planar. In fact, even deciding whether the number of satisfying assignments is even (i.e., counting *modulo two*) in conjunctive normal form formulas where each variable appears at most twice, each clause has at most three variables, and the incidence graph (where nodes are variables and clauses, and edges connect variables appearing in clauses) of the formula is planar is known to be  $NP$ -hard by a randomized reduction [140]. Interestingly, counting the number of satisfying assignments *modulo seven* (!) of that same class of formulas is polynomial-time computable [132].

In this appendix, we present another class of tractable model counting problems defined by its graphical representation. In particular, we develop a polynomial-time algorithm for formulas in monotone conjunctive normal form whose clauses can be partitioned into two sets such that (i) any two clauses in the same set have the same number of variables which are not shared between them, and (ii) any two clauses in different sets share exactly one variable. These formulas lead to intersection graphs (where nodes are clauses, and edges connect clauses which share variables) which are bipartite complete. We state our result in the language of edge coverings; the use of a graph problem makes communication easier with no loss of generality.

The basics of model counting and the particular class of problems we consider are presented in B.1. We then examine the problem of counting edge covers in black-and-white graphs in B.2, and describe a polynomial-time algorithm for counting edge covers of a certain class of black-and-white graphs in B.3. Restrictions are removed in B.4, and we comment on possible extensions of the algorithms in B.5.

## B.1 Model counting: some needed concepts

Say that two clauses do not intersect if the variables in one clause do not appear in the other. If  $X$  is the largest set of variables that appear in two clauses, we say that the clauses intersect (at  $X$ ). For instance, the clauses  $X_1 \vee X_2 \vee X_3$  and  $\neg X_2 \vee \neg X_4$  intersect at  $\{X_2\}$ . A clause containing  $k$  variables is called a  $k$ -clause, and  $k$  is called the size of the clause. The degree of a variable in a CNF formula is the number of clauses in which either the variable or its negation appears. A CNF formula where every variable has degree at most two is said *read-twice*. If any two clauses intersect in at most one variable, the formula is said *linear*. The formula  $(X_1 \vee X_2) \wedge (\neg X_1 \vee \neg X_3)$  is a linear read-twice 2CNF containing two 2-clauses that intersect at  $X_1$ . The degree of  $X_1$  is two, while the degree of either  $X_2$  or  $X_3$  is one. To recap, a formula is *monotone* if no variable appears negated, such as in  $X_1 \vee X_2$ .

We can graphically represent the dependencies between variables and clauses in a CNF formula in many ways. The *incidence graph* of a CNF formula is the bipartite graph with variable-nodes and clause-nodes. The variable-nodes correspond to variables of the formula, while the clause-nodes correspond to clauses. An edge is drawn between a variable-node and a clause-node if and only if the respective variable appears in the respective clause. The *primal graph* of a CNF formula is a graph whose nodes are variables and edges connect variables that co-appear in some clause. The primal graph can be obtained from the incidence graph by deleting clause-nodes (along with their edges) and pairwise connecting their neighbors. The *intersection graph* of a CNF formula is the graph whose nodes correspond to clauses, and an edge connects two nodes if and only if the corresponding clauses

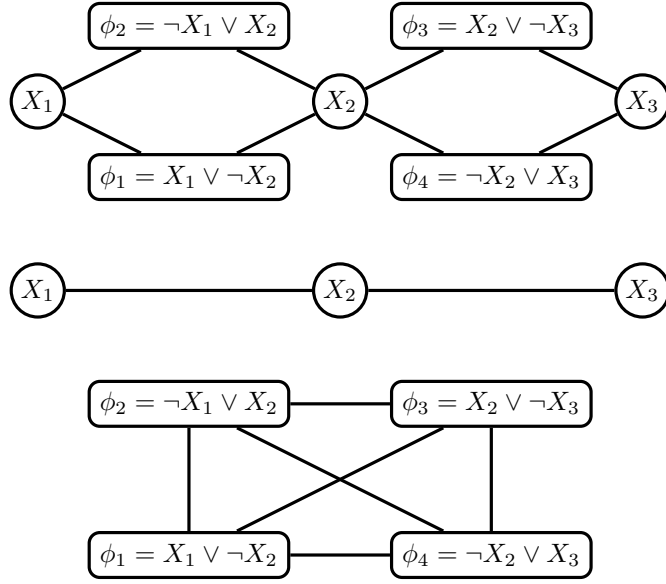


Figure 17: Graphical illustrations of the formula  $(X_1 \vee \neg X_2) \wedge (\neg X_1 \vee X_2) \wedge (X_2 \vee \neg X_3) \wedge (\neg X_2 \vee X_3)$ . Top: incidence graph. Middle: primal graph. Bottom: intersection graph.

intersect. The intersection graph can be obtained from the incidence graph by deleting variable-nodes and pairwise connecting their neighbors. Figure 17 shows examples of graphical illustrations of a Boolean formula. We represent clauses as rectangles and variables as circles.

A CNF formula  $\phi$  is satisfied by an assignment  $\sigma$  (written  $\sigma \models \phi$ ) if each clause contains either a nonnegated variable  $X_i$  such that  $\sigma(X_i) = 1$  or a negated variable  $X_j$  such that  $\sigma(X_j) = 0$ . In this case, we say that  $\sigma$  is a model of  $\phi$ . For monotone CNF formulas, this condition simplifies to the existence of a variable  $X_i$  in each clause for which  $\sigma(X_i) = 1$ . Hence, monotone formulas are always satisfiable (by the trivial model that assigns every variable the value one). The *model count* of a formula  $\phi$  is the number  $Z(\phi) = |\{\sigma : \sigma \models \phi\}|$  of models of the formula. The *model counting problem* is to compute the model count of a given CNF formula  $\phi$ .

In this appendix, we consider linear monotone CNF formulas whose intersection graph is bipartite complete, and such that all clauses in the same part have the same size. These assumptions imply that each variable appears in at most two clauses (hence the formula is read-twice). We call CNF formulas satisfying all of these assumptions linear monotone clause-bipartite complete (LinMonCBPC) formulas. Under these assumptions, we show that model counting can be performed in quadratic time in the size of the input. It is our hope that in future work some of these assumptions can be relaxed. However, due to the results mentioned previously, we do not expect that much can be relaxed without moving to #P-completeness.

The set of model counting problems generated by LinMonCBPC formulas is equivalent to the following problem. Take integers  $m, n, M, N$  such that  $N > n > 0$  and  $M > m > 0$ , and compute how many  $\{0, 1\}$ -valued matrices of size  $M$ -by- $N$  exist such that (i) each of the first  $m$  rows has at least one cell with value one, and (ii) each of the first  $n$  columns has at least one cell with value one. Call  $A_{ij}$  the value of the  $i$ th row,  $j$ th column. The problem is equivalent to computing the number of matrices  $A_{M \times N}$  with  $\sum_{j=1}^N A_{ij} > 0$ , for  $i = 1, \dots, m$ , and  $\sum_{i=1}^M A_{ij} > 0$ , for  $j = 1, \dots, n$ . This problem can be



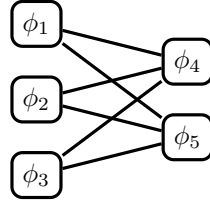


Figure 18: Intersection graph for the LinMonCBPC formula described in the text.

encoded as the model count of the CNF formula whose clauses are

$$\begin{aligned}
& A_{11} \vee A_{12} \vee \cdots \vee A_{1n} \vee \cdots \vee A_{1N}, \\
& A_{21} \vee A_{22} \vee \cdots \vee A_{2n} \vee \cdots \vee A_{2N}, \\
& \vdots \\
& A_{m1} \vee A_{n2} \vee \cdots \vee A_{mn} \vee \cdots \vee A_{1N}, \\
& A_{11} \vee A_{21} \vee \cdots \vee A_{m1} \vee \cdots \vee A_{M1}, \\
& \vdots \\
& A_{1n} \vee A_{2n} \vee \cdots \vee A_{mn} \vee \cdots \vee A_{MN}.
\end{aligned}$$

The first  $m$  clauses are the row constraints, while the last  $n$  clauses are the columns constraints. The row constraints have size  $n$ , and the column constraints have size  $m$ . The  $i$ th row constraint intersects with the  $j$ th column constraint at the variable  $A_{ij}$ . For example, given integers  $m = 3, n = 2, M = 5, N = 6$ , the equivalent model counting problem has clauses

$$\begin{aligned}
\phi_1 & : A_{11} \vee A_{12} \vee A_{13} \vee A_{14} \vee A_{15} \vee A_{16}, \\
\phi_2 & : A_{21} \vee A_{22} \vee A_{23} \vee A_{24} \vee A_{25} \vee A_{26}, \\
\phi_3 & : A_{31} \vee A_{32} \vee A_{33} \vee A_{34} \vee A_{35} \vee A_{36}, \\
\phi_4 & : A_{11} \vee A_{21} \vee A_{31} \vee A_{41} \vee A_{51}, \\
\phi_5 & : A_{12} \vee A_{22} \vee A_{32} \vee A_{42} \vee A_{52}.
\end{aligned}$$

The intersection graph of that formula is show in Figure 18. Note that for the complexity of both problems be equivalent we must have the integers in the matrix problem be given in unary notation (otherwise building the equivalent formula takes time exponential in the input).

## B.2 Counting edge covers and its connection to model counting

A *black-and-white graph* (bw-graph) is a triple  $G = (V, E, \chi)$  where  $(V, E)$  is a simple undirected graph and  $\chi : V \rightarrow \{0, 1\}$  is binary valued function on the node set (assume 0 means white and 1 means black).<sup>9</sup> We denote by  $E_G(u)$  the set of edges incident in a node  $u$ , and  $N_G(u)$  the open neighborhood of  $u$  (i.e., not including  $u$ ). Let  $G = (V, E, \chi)$  be a bw-graph. An edge  $e = (u, v) \in E$  can be classified into one of three categories:<sup>10</sup>

- **free edge:** if  $\chi(u) = \chi(v) = 0$ ;
- **dangling edge:** if  $\chi(u) \neq \chi(v)$ ; or

<sup>9</sup>In [81] and [82], graphs are uncolored, but edges might contain empty endpoints. These are analogous to white node endpoints in our terminology. We prefer defining coloured graphs and allow only simple edges to make our framework close to standard graph theory terminology.

<sup>10</sup>The classifications of edges given here are analogous to those defined in [81, 82], but not fully equivalent. Regular edges are analogous to the *normal edges* defined in [81, 82].

- **regular edge:** if  $\chi(u) = \chi(v) = 1$ .

In the graph in Figure 15(b), the edge  $(f, g)$  is a dangling edge while the edge  $(g, j)$  is a free edge. The edge  $(f, g)$  in the graph in Figure 15(a) is a regular edge.

An *edge cover* of a bw-graph  $G$  is a set  $C \subseteq E$  such that for each node  $v \in V$  with  $\chi(v) = 1$  there is *at least one* edge  $e \in C$  incident in it. An edge cover for the graph in Figure 15(a) is  $\{(a, d), (d, g), (e, g), (f, g), (h, j)\}$ . We denote by  $Z(G)$  the number of edge covers of a bw-color graph  $G$ . Computing  $Z(G)$  is #P-complete [18], and admits an FPTAS [81, 82].

Consider a LinMonCBPC formula and let  $(L, R, E_{LR})$  be its intersection graph, where  $L$  and  $R$  are the two partitions. Call  $s_L$  and  $s_R$  the sizes of a clause in  $L$  and  $R$ , respectively (by construction, all clauses in the same part have the same size), and let  $k_L = s_L - |R|$  and  $k_R = s_R - |L|$ . The value of  $k_L + k_R$  is the number of variables that appear in a single clause. Since the graph is bipartite complete,  $k_L, k_R \geq 0$ . Obtain a bw-graph  $G = (V_1 \cup V_2 \cup V_3 \cup V_4, E, \chi)$  such that

1.  $V_1 = \{1, \dots, k_L\}$ ,  $V_2 = L$ ,  $V_3 = R$  and  $V_4 = \{1, \dots, k_R\}$ ;
2. All nodes in  $V_1 \cup V_4$  are white, and all nodes in  $V_2 \cup V_3$  are black;
3. There is an edge connecting  $(u, v)$  in  $E$  for every  $u \in V_1$  and  $v \in V_2$ , for every  $(u, v) \in E_{LR}$ , and for every  $u \in V_3$  and  $v \in V_4$ .

We call  $\mathcal{B}$  the family of graphs that can be obtained by the procedure above. Figure 15(a) depicts an example of a graph in  $\mathcal{B}$  obtained by applying the procedure to the formula represented in the Figure 18. By construction, for any two nodes  $u, v \in V_i$ ,  $i = 1, \dots, 4$ , it follows that  $N_G(u) = N_G(v)$  and  $(u, v) \notin E$ . The following result shows the equivalence between edge covers and model counting.

**Proposition 2.** *Consider a LinMonCBPC formula  $\phi$  and suppose that  $G = (V_1, V_2, V_3, V_4, E, \chi)$  is a corresponding bw-graph in  $\mathcal{B}$ . Then number of edge covers of  $G$  equals the model counting of  $\phi$ , that is,  $Z(G) = Z(\phi)$ .*

*Proof.* Let  $u_i$  denote the node in  $G$  corresponding to a clause  $\phi_i$  in  $\phi$ . Label each edge  $(u_i, v_j)$  for  $\phi_i \in L$  and  $\phi_j \in R$  with the variable corresponding to the intersection of the two clauses. For each  $\phi_i \in L$ , label each dangling edge  $(u, u_i)$  incident in  $u_i$  with a different variable that appears only at  $\phi_i$ . Similarly, label each dangling edge  $(u_j, u)$  with a different variable that appears only at  $\phi_j \in R$ . Note that the labeling function is bijective, as every variable in  $\phi$  labels exactly one edge of  $G$ .

Now consider a satisfying assignment  $\sigma$  of  $\phi$  and let  $C$  be set of edges labeled with the variables  $X_i$  such that  $\sigma(X_i) = 1$ . Then  $C$  is an edge cover since every clause (node in  $G$ ) has at least one variable (incident edge) with  $\sigma(X_i) = 1$  and the corresponding edge is in  $C$ . To show the converse holds, consider an edge cover  $C$  for  $G$ , and construct an assignment such that  $\sigma(X_i) = 1$  if the edge labeled by  $X_i$  is in  $C$  and  $\sigma(X_i) = 0$  otherwise. Then  $\sigma$  satisfies  $\phi$ , since for every clause  $\phi_i$  (node  $u_i$ ) there is a variable in  $\phi_i$  with  $\sigma(X_i)$  (incident edge in  $u_i$  in  $C$ ). Since there are as many edges as variables, the correspondence between edge covers and satisfying assignment is one-to-one.  $\square$

### B.3 A dynamic programming approach to counting edge covers

In this section we derive an algorithm for computing the number of edge covers of a graph in  $\mathcal{B}$ . Let  $e$  be an edge and  $u$  be a node in  $G = (V, E, \chi)$ . We define the following operations and notation:

- **edge removal:**  $G - e = (V, E \setminus \{e\}, \chi)$ .
- **node whitening:**  $G - u = (V, E, \chi')$ , where  $\chi'(u) = 0$  and  $\chi'(v) = \chi(v)$  for  $v \neq u$ .

Note that these operations do not alter the node set, and that they are associative (e.g.,  $G - e - f = G - f - e$ ,  $G - u - v = G - v - u$ , and  $G - e - u = G - u - e$ ). Hence, if  $E = \{e_1, \dots, e_d\}$  is a set of edges, we can write  $G - E$  to denote  $G - e_1 - \dots - e_d$  applied in any arbitrary order. The same is true

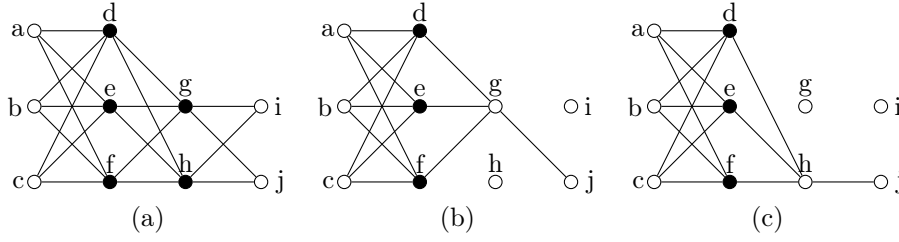


Figure 19: (a) A graph  $G$  in  $\mathcal{B}$ . (b) The graph  $G - E_G(h) - (i, g) - h - g$ . (c) The graph  $G - E_G(g) - (i, h) - g - h$ .

for node whitening and for any combination of node whitening and edge removal. These operations are illustrated in the examples in Figure 15.

The following result shows that the number of edge covers can be computed recursively on smaller graphs:

**Proposition 3.** *Let  $e = (u, v)$  be a dangling edge with  $u$  colored black. Then:*

$$Z(G) = 2Z(G - e - u) - Z(G - E_G(u) - u).$$

*Proof.* There are  $Z(G - e - u)$  edge covers of  $G$  that contain  $e$  and  $Z(G - e)$  edge covers that do not contain  $e$ . Hence,  $Z(G) = Z(G - e - u) + Z(G - e)$ . Now, consider the graph  $G' = G - e - u$ . There are  $Z(G - e)$  edge covers of  $G'$  that contain at least one edge of  $E_{G'}(u)$  and  $Z(G - E_G(u) - u)$  edge covers that contain no edge of  $E_{G'}(u)$ . Thus  $Z(G - e - u) = Z(G - e) + Z(G - E_G(u) - u)$ . Substituting  $Z(G - e)$  in the first identity gives us the desired result.  $\square$

Free edges and isolated white nodes can be removed by adjusting the edge count correspondingly:

**Proposition 4.** *We have:*

1. *Let  $e = (u, v)$  be a free edge of  $G$ . Then  $Z(G) = 2Z(G - e)$ .*
2. *If  $u$  is an isolated white node (i.e.,  $N_G(u) = \emptyset$ ) then  $Z(G) = Z(G - u)$ .*

*Proof.* (1) If  $C$  is an edge cover of  $G - e$  then both  $C$  and  $C \cup \{e\}$  are edge covers of  $G$ . Hence, the number of edge covers containing  $e$  equals the number  $Z(G - e)$  of edge covers not containing  $e$ . (2) Every edge cover of  $G$  is also an edge cover of  $G - u$  and vice-versa.  $\square$

We can use the formulas in Propositions 3 and 4 to compute the edge cover count of a graph recursively. Each recursion computes the count as a function of the counts of two graphs obtained by the removal of edges and whitening of nodes. Such a naive approach requires an exponential number of recursions (in the number of edges or nodes of the initial graph) and finishes after exponential time. We can transform such an approach into a polynomial-time algorithm by exploiting the symmetries of the graphs produced during the recursions. In particular, we take advantage of the invariance of edge cover count to isomorphisms of a graph, as we discuss next.

We say that two bw-graphs  $G = (V, E, \chi)$  and  $G' = (V', E', \chi')$  are *isomorphic* if there is a bijection  $\gamma$  from  $V$  to  $V'$  (or vice-versa) such that (i)  $\chi(v) = \chi'(\gamma(v))$  for all  $v \in V$ , and (ii)  $(u, v) \in E$  if and only if  $(\gamma(u), \gamma(v)) \in E'$ . In other words, two bw-graphs are isomorphic if there is a color-preserving renaming of nodes that preserves the binary relation induced by  $E$ . The function  $\gamma$  is called an *isomorphism* from  $V$  to  $V'$ . The graphs in Figures 15(b) and 15(c) are isomorphic by an isomorphism that maps  $g$  in  $h$  and maps any other node into itself. If  $C$  is an edge cover of  $G$  and  $\gamma$  is an isomorphism between  $G$  and  $G'$ , then  $C' = \{(\gamma(u), \gamma(v)) : (u, v) \in C\}$  is an edge cover for  $G'$  and vice-versa. Hence,  $Z(G) = Z(G')$ . The following result shows how to obtain isomorphic graphs with a combination of node whitenings and edge removals.

**Proposition 5.** Consider a bw-graph  $G$  with nodes  $v_1, \dots, v_n$ , where  $N_G(v_1) = \dots = N_G(v_n) \neq \emptyset$  and  $\chi_G(v_1) = \dots = \chi_G(v_n)$ . For any node  $w \in N_G(v_1)$ , mapping  $\gamma : \{v_1, \dots, v_n\} \rightarrow \{v_1, \dots, v_n\}$ , and nonnegative integers  $k_1$  and  $k_2$  such that  $k_1 + k_2 \leq n$  the graphs  $G' = G - E_G(v_1) - \dots - E_G(v_{k_1}) - (w, v_{k_1+1}) - \dots - (w, v_{k_1+k_2}) - v_1 - \dots - v_{k_1+k_2}$  and  $G'' = G - E_G(\gamma(v_1)) - \dots - E_G(\gamma(v_{k_1})) - (w, \gamma(v_{k_1+1})) - \dots - (w, \gamma(v_{k_1+k_2})) - \gamma(v_1) - \dots - \gamma(v_{k_1+k_2})$  are isomorphic.

*Proof.* Let  $\gamma'$  be the bijection on the nodes of  $G$  that extends  $\gamma$ , that is,  $\gamma'(u) = u$  for  $u \notin \{v_1, \dots, v_n\}$  and  $\gamma'(u) = \gamma(v_i)$ , for  $i = 1, \dots, n$ . We will show that  $\gamma'$  is an isomorphism from  $G'$  to  $G''$ . First note that  $\chi_G(u) = \chi_G(\gamma(u))$  for every node  $u$ . The only nodes that have their color (possibly) changed in  $G'$  with respect to  $G$  are the nodes  $v_1, \dots, v_{k_1+k_2}$ , and these are white nodes in  $G'$ . Likewise, the only nodes that would (possibly) changed color in  $G''$  were  $\gamma(v_1), \dots, \gamma(v_{k_1+k_2})$  and these are white in  $G''$ . Hence,  $\chi_{G'}(u) = \chi_{G''}(\gamma(u))$  for every node  $u$ .

Now let us look at the edges. First note that since  $N_G(v_i)$  is constant through  $i = 1, \dots, n$ ,  $G'$  and  $G''$  have the same number of edges. Hence, it suffices to show that for each edge  $(u, v)$  in  $G'$  the edge  $(\gamma'(u), \gamma'(v))$  is in  $G''$ . The only edges modified in obtaining  $G'$  and  $G''$  are, respectively, those incident in  $v_1, \dots, v_{k_1+k_2}$  and in  $\gamma(v_1), \dots, \gamma(v_{k_1+k_2})$ . Consider an edge  $(u, v)$  where  $u, v \notin \{v_1, \dots, v_n\}$  (hence not in  $E_G(v_i)$  for any  $i$ ). If  $(u, v) = (\gamma'(u), \gamma'(v))$  is in  $G'$  then it is also in  $G''$ . Now consider an edge  $(u, v_i)$  in  $G$  where  $u \notin \{w, v_{k_1+1}, \dots, v_n\}$  and  $k_1 < i \leq k_1 + k_2$ . Then  $(u, v_i)$  is in  $G'$  and  $(\gamma'(u), \gamma'(v_i))$  is in  $G''$ . Note that  $u$  could be in  $N_G(v_i)$  for  $k_1 + k_2 < i \leq n$ .  $\square$

According to the proposition above, the graphs in Figures 15(b) and 15(c) are isomorphic by a mapping from  $g$  to  $h$  (and with  $w = i$ ). Hence, the number of edge covers in either graph is the same.

The algorithms `RightRecursion` and `LeftRecursion` described in Figures 20 and 21, respectively, exploit the isomorphisms described in Proposition 5 in order to achieve polynomial-time behavior when using the recursions in Propositions 3 and 4. Either algorithm requires a base white node  $w$  and integers  $k_1$  and  $k_2$  specifying the recursion level (with the same meaning as in Proposition 5). Unless  $k_1 + k_2$  equals the number of neighbors of  $w$  in the original graph, a call to either algorithm generates two more calls to the same algorithm: one with the graph obtained by removing edge  $(w, v_h)$  and whitening  $v_h$ , and another by removing edges  $E(v_h)$  and whitening  $v_h$ . Assume that  $|V_2| \geq |V_3|$  (if  $|V_3| > |V_2|$  we can simply manipulate node sets to obtain an isomorphic graph satisfying the assumption). The `RightRecursion` algorithm first checks whether the value for the current recursion level has been already computed; if yes, then it simply returns the cached value; otherwise it uses the formula in Proposition 3 (and possibly the isomorphism in Proposition 5) and generates two calls of the same algorithm on smaller graphs (i.e. with fewer edges) to compute the edge cover counting for the current graph and stores the result in memory. The recursion continues until the recursion levels equates with the number of nodes in  $V_3$ , in which case it checks for free edges and isolated nodes, removes them and computes the correction factor  $2^k$ , where  $k$  is the number of free edges, and calls the algorithm `LeftRecursion` to start a new recursion. At this point the graph in the input is bipartite complete and contains only nodes in  $V_1$  and  $V_2$ . The latter algorithm behaves very similarly to the former except at the termination step. When all neighbors  $v_h$  of  $w$  have been whitened the graph no longer contains black nodes, and the corresponding edge cover count can be directly computed using the formulas in Proposition 4. Note that a different cache function must be used when we call `LeftRecursion` from `RightRecursion` (this can be done by instantiating an object at that point and passing it as argument; we avoid stating the algorithm is this way to avoid cluttering).

Note that the algorithms do not use the color of nodes, which hence does not need to be stored or manipulated. In fact the node whitening operations ( $-v_h$  or  $-u_h$ ) performed when calling the recursion are redundant and can be neglected without altering the soundness of the procedure (we decided to leave these operations as they make the connection with Proposition 3 more clear).

Figure 22 shows the recursion diagram of a run of `RightRecursion`. Each box in the figure represents a call of the algorithm with the graph drawn as input. The left child of each box is the call `RightRecursion( $G - (v_h, w) - v_h, w, k_1, k_2 + 1$ )`, and the right child is the call `RightRecursion( $G - E_G(v_h) - v_h, w, k_1 + 1, k_2$ )`. For instance, the topmost box represents `RightRecursion( $G_0, w, 0, 0$ )`, which computes

```

1: if Cache( $w, k_1, k_2$ ) > 0 then
2:   return Cache( $w, k_1, k_2$ )
3: else
4:   if  $k_1 + k_2 < m$  then
5:     Let  $h \leftarrow k_1 + k_2 + 1$ 
6:     Cache( $w, k_1, k_2$ )  $\leftarrow 2 \times \text{RightRecursion}(G - (v_h, w) - v_h, w, k_1, k_2 + 1) - \text{RightRecursion}(G -$ 
        $E_G(v_h) - v_h, w, k_1 + 1, k_2)$ 
7:     return Cache( $w, k_1, k_2$ )
8:   else
9:     Let  $k = |\{(u, v) : u \in V_4\}|$  be the number of free edges
10:    Remove any edges with an endpoint in  $V_4$  and all the resulting isolated nodes
11:    Set  $V_1 \leftarrow V_1 \cup V_3, V_3 \leftarrow \emptyset$ 
12:    if  $V_1$  is empty then
13:      return 0
14:    end if
15:    Select an arbitrary  $w' \in V_1$ 
16:    return  $2^k \times \text{LeftRecursion}(G, w', 0, 0)$ 
17:  end if
18: end if

```

Figure 20: Algorithm RightRecursion: Takes a graph  $G = (V_1, V_2, V_3, V_4, E)$  with  $V_3 = \{v_1, \dots, v_m\}$ ,  $m > 0$ , a node  $w \in V_4$ , and nonnegative integers  $k_1$  and  $k_2$ ; outputs  $Z(G)$ .

```

1: if Cache( $w, k_1, k_2$ ) is undefined then
2:   if  $k_1 + k_2 < n$  then
3:     Let  $h \leftarrow k_1 + k_2 + 1$ 
4:     Cache( $w, k_1, k_2$ )  $\leftarrow 2 \times \text{LeftRecursion}(G - (u_h, w) - u_h, k_1, k_2 + 1) - \text{LeftRecursion}(G -$ 
        $E_G(u_h) - u_h, k_1 + 1, k_2)$ 
5:   else
6:     Cache( $w, k_1, k_2$ )  $\leftarrow 2^{|E|}$ 
7:   end if
8: end if
9: return Cache( $w, k_1, k_2$ )

```

Figure 21: Algorithm LeftRecursion: Takes a bipartite graph  $G = (V_1, V_2, E)$  with  $V_2 = \{u_1, \dots, u_n\}$ ,  $n > 0$ , a node  $w \in V_1$ , nonnegative integers  $k_1$  and  $k_2$ ; outputs  $Z(G)$ .

$Z(G_0)$  as the sum of  $2Z(G_1)$  and  $-Z(G_{24})$ , which are obtained, respectively, from the calls corresponding to its left and right children. The number of the graph in each box corresponds to the order in which each call was generated. Solid arcs represent non cached calls, while dotted arcs indicate cached calls. For instance, by the time  $\text{RightRecursion}(G_{24}, w, 1, 0)$  is called,  $\text{RightRecursion}(G_{13}, w, 0, 0)$  has already been computed so the value of  $Z(G_{13})$  is simply read from memory and returned. When called in the graph in the top, with to rightmost node  $w$ , and integers  $k_1 = k_2 = 0$ , the algorithm computes the partition function  $Z(G_0)$  as the sum of  $2Z(G_1)$  and  $-Z(G_{24})$ , where  $G_1$  is obtained from the removal of edge  $(v_1, w)$  and whitening of  $v_1$ , while  $G_{24}$  is obtained by removing edges  $E_{G_1}(v_1)$  and whitening of  $v_1$ . The recursion continues until all incident edges on  $w$  have been removed, at which point it removes free edges and isolated nodes and calls  $\text{LeftRecursion}$ . The recursion diagram for the call of  $\text{LeftRecursion}(G_4, w, 0, 0)$  where  $w$  is the top leftmost node of  $G_4$  in the figure is shown in Figure 23. The semantics of the diagram is analogous. Note that the recursion of  $\text{LeftRecursion}$  eventually reaches a graph with no black nodes, for which the edge cover count can be directly computed (in closed-form).

In these diagrams, it is possible to see how the isomorphisms stated in Proposition 5 are used by the algorithms and lead to polynomial-time behavior. For instance, in the run in Figure 22, the graph  $G_{13}$  is not the graph obtained from  $G_{24}$  by removing edge  $(v_2, w)$  and whitening  $v_2$  but instead is isomorphic to it. Note that both  $G_{13}$  and its isomorphic graph obtained as the left child of  $G_{24}$  were obtained by one operation of edge removal  $-(w, v_i)$  and one operation of neighborhood removal  $-E(v_i)$ , plus node whitenings of  $v_1$  and  $v_2$ . Hence, Proposition 5 guarantees their isomorphism.

The polynomial-time behavior of the algorithms strongly depends on caching the calls (dotted arcs) and exploiting known isomorphisms. For instance, in the run in Figure 22, the graph  $G_{13}$  is not the graph obtained from  $G_{24}$  by removing edge  $(v_2, w)$  and whitening  $v_2$  but instead is isomorphic to it. Note that both  $G_{13}$  and its isomorphic graph obtained as the left child of  $G_{24}$  were obtained by one operation of edge removal  $(w, v_i)$  and one operation of neighborhood removal  $E(v_i)$ , plus node whitenings of  $v_1$  and  $v_2$ . Hence, Proposition 5 guarantees their isomorphism.

Without the caching of computations, the algorithm would perform exponentially many recursive calls (and its corresponding diagram would be a binary tree with exponentially many nodes). The use of caching allows us to compute only one call of  $\text{RightRecursion}$  for each configuration of  $k_1, k_2$  such that  $k_1 + k_2 \leq n$ , resulting in at most  $\sum_{i=0}^n (i+1) = (n+1)(n+2)/2 = O(n^2)$  calls for  $\text{RightRecursion}$ , where  $n = |V_3|$ . Similarly, each call of  $\text{LeftRecursion}$  requires at most  $\sum_{i=0}^m (i+1) = (m+1)(m+2)/2 = O(m^2)$  recursive calls for  $\text{LeftRecursion}$ , where  $m = |V_2|$ . Each call to  $\text{RightRecursion}$  with  $k_1 + k_2 = n$  generates a call to  $\text{LeftRecursion}$  (there are  $n + 1$  such configurations). Hence, the overall number of recursions (i.e., call to either function) is

$$\frac{(n+1)(n+2)}{2} + (n+1)\frac{(m+1)(m+2)}{2} = O(n^2 + n \cdot m^2).$$

This leads us to the following result.

**Theorem 24.** *Let  $G$  be a graph in  $\mathcal{B}$  with  $w \in V_4 \neq \emptyset$ . Then the call  $\text{RightRecursion}(G, w, 0, 0)$  outputs  $Z(G)$  in time and memory at most cubic in the number of nodes of  $G$ .*

*Proof.* Except when  $k_1 + k_2 = n$ ,  $\text{RightRecursion}$  calls the recursion given in Proposition 3 with the isomorphisms in Proposition 5 (any graph obtained from  $G$  by  $k_1$  operations  $-E_G(v_i)$  and  $k_2$  operations  $-(w, v_i)$  are isomorphic). For  $k_1 + k_2$ , any edge left connecting a node in  $V_3$  and a node in  $V_4$  must be a free edge (since all nodes in  $V_4$  have been whitened), hence they can be removed according to Proposition 4 with the appropriate correction of the count. By the same result, any isolated node can be removed. When the remaining nodes in  $V_3$  are transferred to  $V_1$ , the resulting graph is bipartite complete (with white nodes in one part and black nodes in the other). Hence, we can call  $\text{LeftRecursion}$ , which is guaranteed to compute the correct count by the same arguments.

The cubic time and space behavior is due to  $\text{RightRecursion}$  and  $\text{LeftRecursion}$  being called at most  $O(n^2)$  and  $O(nm^2)$ , respectively, and by the fact that each call consists of local operations (edge removals and node whitenings) which take at most linear time in the number of nodes and edges of the graph.  $\square$

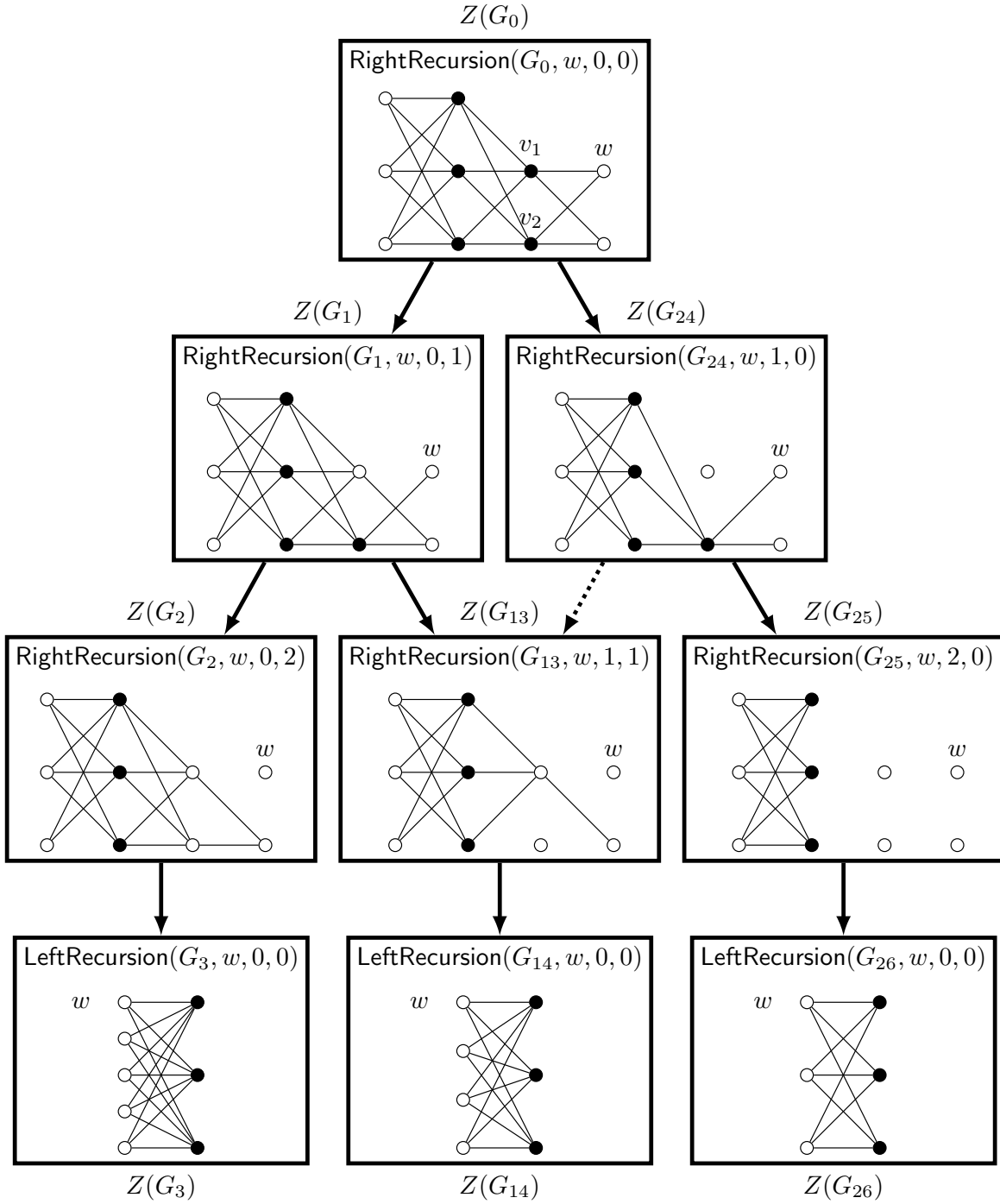


Figure 22: Simulation of  $\text{RightRecursion}(G_0, w, 0, 0)$ .

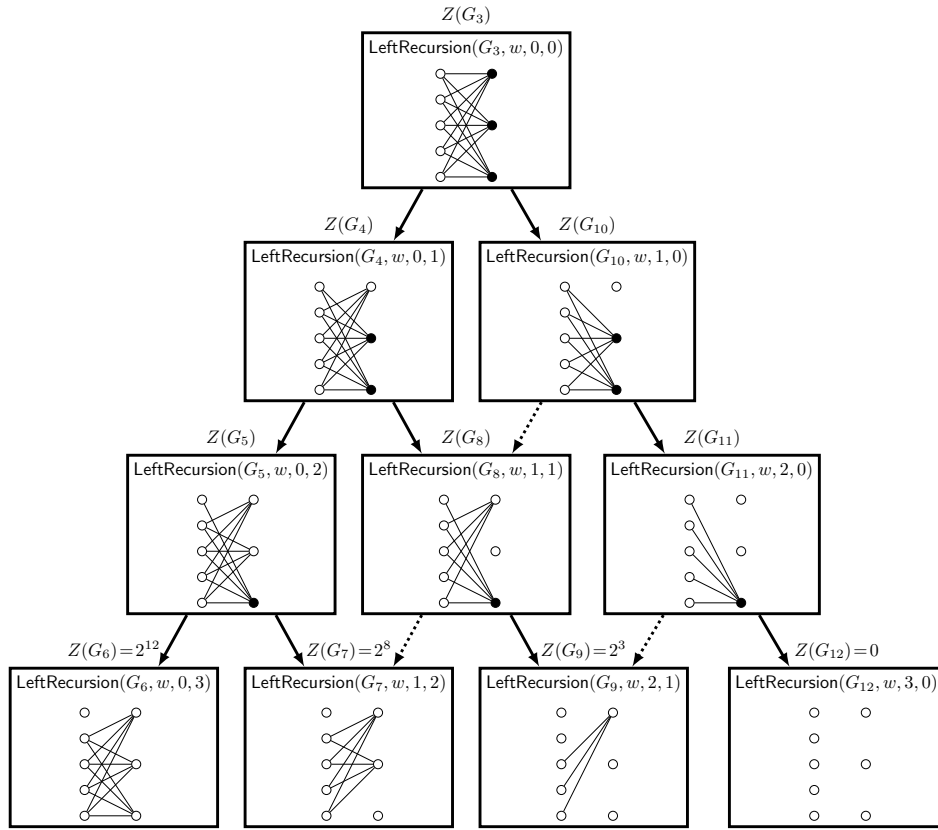


Figure 23: Simulation of  $\text{LeftRecursion}(G_3, w, 0, 0)$ .



## B.4 Graphs with no dangling edges

The algorithm `RightRecursion` requires the existence of a dangling edge. Now it might be that the graph contains no white nodes (hence no dangling edges), that is, that  $G$  is bipartite complete graph for  $V_2 \cup V_3$ . The next result shows how to decompose the problem of counting edge covers in smaller graphs that either contain dangling edges, or are also bipartite complete.

**Proposition 6.** *Let  $G$  be a bipartite complete bw-graph with all nodes colored black and  $e = (u, v)$  be some edge. Then  $Z(G) = 2Z(G - e - u - v) - Z(G - E_G(v) - v) - Z(G - E_G(u) - u) - Z(G - E_G(u) - E_G(v) - u - v)$ .*

*Proof.* The edge covers of  $G$  can be partitioned according to whether they contain the edge  $e$ . The number of edge covers that contain  $e$  is not altered if we color both  $u$  and  $v$  white. Thus,  $Z(G) = Z(G - e - u - v) + Z(G - e)$ . Let  $e_1, \dots, e_n$  be the edges incident in  $u$  other than  $e$ , and  $f_1, \dots, f_m$  be the edges incident in  $v$  other than  $v$ . We have that  $Z(G - e - u - v) = Z(G - e - u - v) + Z(G - E_G(u) - u) + Z(G - e) + Z(G - E_G(u) - E_G(v) - u - v)$ . Substituting  $Z(G - e)$  into the first equation obtains the result.  $\square$

In the result above, the graphs  $G - e - u - v$ ,  $G - E_G(v) - v$  and  $G - E_G(u) - u$  are in  $\mathcal{B}$  and contain dangling edges, while the graph  $G - E_G(u) - E_G(v) - u - v$  is bipartite complete. Note that Proposition 5 can be applied to show that altering the edges on which the operations are applied lead to isomorphic graphs. A very similar algorithm to `LeftRecursion`, implementing the recursion in the result above in polynomial-time can be easily derived.

## B.5 Extensions

Previous results can be used beyond the class of graphs  $\mathcal{B}$ . For instance, the algorithms can compute the edge cover count for any graph that can be obtained from a graph  $G$  in  $\mathcal{B}$  by certain sequences of edge removals and node whitenings, which includes graphs not in  $\mathcal{B}$ . Graphs that satisfy the properties of the class  $\mathcal{B}$  except that every node in  $V_2$  (or  $V_4$  or both) are pairwise connected can also have their edge cover count computed by the algorithm (as this satisfies the conditions in Proposition 5). Another possibility is to consider graphs which can be decomposed in graphs  $\mathcal{B}$  by polynomially many applications of Proposition 3.

We can also consider more general forms of counting problems. A simple mechanism for randomly generating edge covers is to implement a Markov Chain with starts with some trivial edge cover (e.g. one containing all edges) and moves from an edge cover  $X_t$  to an edge cover  $X_{t+1}$  by the following Glauber Dynamics-type move: (1) Select an edge  $e$  uniformly at random; (2a) if  $e \notin X_t$ , make  $X_{t+1} = X_t \cup \{e\}$  with probability  $\lambda/(1 + \lambda)$ ; (2b) if  $e \in X_t$  and if  $X_t \setminus \{e\}$  is an edge cover, make  $X_{t+1} = X_t \setminus \{e\}$  with probability  $1/(1 + \lambda)$ ; (2c) else make  $X_{t+1} = X_t$ . The above Markov chain can be shown to be ergodic and to converge to a stationary distribution which samples an edge cover  $C$  with probability  $\lambda^{|C|}$  [13, 11]. When  $\lambda = 1$ , the algorithm performs uniform sampling of edge covers. A related problem is to compute the total probability mass that such an algorithm will assign to sets of edge covers given a bw-graph  $G$ , the so-called *partition function*:  $Z(G, \lambda) = \sum_{C \in \text{EC}(G)} \lambda^{|C|}$ , defined for any real  $\lambda > 0$ , where  $\text{EC}(G)$  is the set of edge covers of  $G$ . For  $\lambda = 1$  the problem is equivalent to counting edge covers. This is also equivalent to weighted model counting of `LinMonCBPC` formulas with uniform weight  $\lambda$ .

The following results are analogous to Propositions 3 and 4 for computing the partition function:

**Proposition 7.** *The following assertions are true:*

1. *Let  $e = (u, v)$  be a free edge of  $G$ . Then  $Z(G) = (1 + \lambda)Z(G - e)$ .*
2. *If  $u$  is an isolated white node (i.e.,  $N_G(u) = \emptyset$ ) then  $Z(G) = Z(G - u)$ .*
3. *Let  $e = (u, v)$  be a dangling edge with  $u$  colored black. Then  $Z(G) = (1 + \lambda)Z(G - e - u) - Z(G - E_G(u) - u)$ .*

Hence, by modifying the weights by which the recursive calls are multiplied, we easily modify algorithms `RightRecursion` and `LeftRecursion` () so as to compute the partition function of graphs in  $\mathcal{B}$  (or equivalently, the partition function of `LinMonCBPC` formulas).

## References

- [1] A. Artale, D. Calvanese, R. Kontchakov, M. Zakharyashev, The *DL-Lite* family and relations, *Journal of Artificial Intelligence Research* 36 (2009) 1–69.
- [2] F. Baader, Terminological cycles in a description logic with existential restrictions, in: *IJCAI*, 2003, pp. 325–330.
- [3] F. Baader, W. Nutt, Basic description logics, in: *Description Logic Handbook*, Cambridge University Press, 2002, pp. 47–100.
- [4] F. Bacchus, *Representing and Reasoning with Probabilistic Knowledge: A Logical Approach*, MIT Press, Cambridge, 1990.
- [5] F. Bacchus, Using first-order probability logic for the construction of Bayesian networks, in: *Conference on Uncertainty in Artificial Intelligence*, 1993, pp. 219–226.
- [6] F. Bacchus, S. Dalmao, T. Pitassi, Solving  $\#SAT$  and Bayesian inference with backtracking search, *Journal of Artificial Intelligence Research* 34 (2009) 391–442.
- [7] D. D. Bailey, V. Dalmau, P. G. Kolaitis, Phase transitions of pp-complete satisfiability problems, *Discrete Applied Mathematics* 155 (2007) 1627–1639.
- [8] M. Bauland, E. Bohler, N. Creignou, S. Reith, H. Schnoor, H. Vollmer, The complexity of problems for quantified constraints, *Theory of Computing Systems* 47 (2010) 454–490.
- [9] P. Beame, G. Van den Broeck, E. Gribkoff, D. Suciu, Symmetric weighted first-order model counting, in: *ACM Symposium on Principles of Database Systems (PODS)*, 2015.
- [10] O. Benjelloun, A. D. Sarma, A. Halevy, M. Theobald, J. Widom, Databases with uncertainty and lineage, *The International Journal on Very Large Data Bases* 17 (2) (2008) 243–264.
- [11] I. Bezáková, W. Rummmler, Sampling edge covers in 3-regular graphs, in: *International Symposium on Mathematical Foundations of Computer Science*, 2009, pp. 137–148.
- [12] D. M. Blei, A. Y. Ng, M. I. Jordan, Latent Dirichlet allocation, *Journal of Machine Learning Research* 3 (2003) 993–1022.
- [13] M. Bordewich, M. Dyer, M. Karpinski, Path coupling using stopping times, in: *International Symposium Fundamentals of Computation Theory*, 2005, pp. 19–31.
- [14] R. Bubley, M. Dyer, Graph orientations with no sink and an approximation for a hard case of  $\#SAT$ , in: *ACM-SIAM Symposium on Discrete Algorithms*, 1997, pp. 248–257.
- [15] H. Buhrman, L. Fortnow, T. Thierauf, Nonrelativizing separations, in: *Proceedings of IEEE Complexity*, 1998, pp. 8–12.
- [16] A. Bulatov, M. Dyer, L. A. Goldberg, M. Jalsenius, M. Jerrum, D. Richerby, The complexity of weighted and unweighted  $\#CSP$ , *Journal of Computer and System Sciences* 78 (2012) 681–688.
- [17] W. L. Buntine, Operations for learning with graphical models, *Journal of Artificial Intelligence Research* 2 (1994) 159–225.

- [18] J.-Y. Cai, P. Lu, M. Xia, Holographic reduction, interpolation and hardness, *Computational Complexity* 21 (4) (2012) 573–604.
- [19] D. Calvanese, G. D. Giacomo, D. Lembo, M. Lenzerini, R. Rosati, DL-Lite: Tractable description logics for ontologies, in: *AAAI*, 2005, pp. 602–607.
- [20] D. Calvanese, G. D. Giacomo, D. Lembo, M. Lenzerini, R. Rosati, Data complexity of query answering in description logics, in: *Knowledge Representation*, 2006, pp. 260–270.
- [21] R. N. Carvalho, K. B. Laskey, P. C. Costa, PR-OWL 2.0 — bridging the gap to OWL semantics, in: *URSW 2008-2010/UniDL 2010, LNAI 7123*, 2013, pp. 1–18.
- [22] I. I. Ceylan, R. Peñaloza, The Bayesian description logic  $\mathcal{BEL}$ , in: *International Joint Conference on Automated Reasoning*, 2014, pp. 480–494.
- [23] M. Chavira, A. Darwiche, On probabilistic inference by weighted model counting, *Artificial Intelligence* 172 (6-7) (2008) 772–799.
- [24] P. C. G. Costa, K. B. Laskey, PR-OWL: A framework for probabilistic ontologies, in: *Conference on Formal Ontology in Information Systems*, 2006.
- [25] V. S. Costa, D. Page, M. Qazi, J. Cussens,  $\text{CLP}(\mathcal{BN})$ : Constraint logic programming for probabilistic knowledge, in: U. Kjaerulff, C. Meek (eds.), *Conference on Uncertainty in Artificial Intelligence*, Morgan-Kaufmann, San Francisco, California, 2003, pp. 517–524.
- [26] F. G. Cozman, D. D. Mauá, Bayesian networks specified using propositional and relational constructs: Combined, data, and domain complexity, in: *AAAI Conference on Artificial Intelligence*, 2015.
- [27] F. G. Cozman, D. D. Mauá, Probabilistic graphical models specified by probabilistic logic programs: Semantics and complexity, in: *Conference on Probabilistic Graphical Models — JMLR Workshop and Conference Proceedings*, vol. 52, 2016, pp. 110–121.
- [28] F. G. Cozman, D. D. Mauá, The structure and complexity of credal semantics, in: *Workshop on Probabilistic Logic Programming*, 2016, pp. 3–14.
- [29] F. G. Cozman, D. D. Mauá, The well-founded semantics of cyclic probabilistic logic programs: meaning and complexity, in: *Encontro Nacional de Inteligência Artificial e Computacional*, 2016, pp. 1–12.
- [30] F. G. Cozman, R. B. Polastro, Complexity analysis and variational inference for interpretation-based probabilistic description logics, in: *Proceedings of the Twenty-Fifth Conference Annual Conference on Uncertainty in Artificial Intelligence (UAI-09)*, AUAI Press, Corvallis, Oregon, 2009, pp. 117–125.
- [31] N. Dalvi, D. Suciu, Efficient query evaluation on probabilistic databases 16 (2007) 523–544.
- [32] N. Dalvi, D. Suciu, The dichotomy of probabilistic inference for unions of conjunctive queries, *Journal of the ACM* 59 (6).
- [33] C. d’Amato, N. Fanizzi, T. Lukasiewicz, Tractable reasoning with Bayesian description logics, in: *International Conference on Scalable Uncertainty Management*, 2008, pp. 146–159.
- [34] A. Darwiche, G. Provan, Query DAGs: A practical paradigm for implementing belief-network inference, in: E. Horvitz, F. Jensen (eds.), *Proceedings of the Twelfth Conference on Uncertainty in Artificial Intelligence*, Morgan Kaufmann, San Francisco, California, United States, 1996, pp. 203–210.

- [35] A. Darwiche, A differential approach to inference in Bayesian networks, *Journal of the ACM* 50 (3) (2003) 280–305.
- [36] A. Darwiche, *Modeling and Reasoning with Bayesian Networks*, Cambridge University Press, 2009.
- [37] A. Darwiche, P. Marquis, A knowledge compilation map, *Journal of Artificial Intelligence Research* 17 (2002) 229–264.
- [38] F. G. C. Denis Deratani Maua, Cassio Polpo de Campos, The complexity of MAP inference in Bayesian networks specified through logical languages, in: *International Joint Conference on Artificial Intelligence*, 2015, pp. 889–895.
- [39] Z. Ding, Y. Peng, R. Pan, BayesOWL: Uncertainty modeling in semantic web ontologies, in: *Soft Computing in Ontologies and Semantic Web*, vol. 204 of *Studies in Fuzziness and Soft Computing*, Springer, Berlin/Heidelberg, 2006, pp. 3–29.
- [40] P. Domingos, W. A. Webb, A tractable first-order probabilistic logic, in: *AAAI*, 2012, pp. 1902–1909.
- [41] A. Durand, M. Hermann, P. G. Kolaitis, Subtractive reductions and complete problems for counting complexity classes, *Theoretical Computer Science* 340 (3) (2005) 496–513.
- [42] D. Fierens, H. Blockeel, M. Bruynooghe, J. Ramon, Logical Bayesian networks and their relation to other probabilistic logical models, in: *Int. Conference on Inductive Logic Programming*, 2005, pp. 121–135.
- [43] D. Fierens, H. Blockeel, J. Ramon, M. Bruynooghe, Logical Bayesian networks, in: *Workshop on Multi-Relational Data Mining*, 2004, pp. 19–30.
- [44] D. Fierens, G. Van den Broeck, J. Renkens, D. Shrerionov, B. Gutmann, G. Janssens, L. de Raedt, Inference and learning in probabilistic logic programs using weighted Boolean formulas, *Theory and Practice of Logic Programming* 15 (3) (2014) 358–401.
- [45] J. Flum, M. Grohe, The parameterized complexity of counting problems, *SIAM Journal of Computing* 33 (4) (2004) 892–922.
- [46] N. Friedman, L. Getoor, D. Koller, A. Pfeffer, Learning probabilistic relational models, in: *International Joint Conference on Artificial Intelligence*, 1999, pp. 1300–1309.
- [47] L. Getoor, N. Friedman, D. Koller, A. Pfeffer, B. Taskar, Probabilistic relational models, in: *Introduction to Statistical Relational Learning*, 2007.
- [48] L. Getoor, J. Grant, PRL: A probabilistic relational language, *Machine Learning* 62 (2006) 7–31.
- [49] L. Getoor, B. Taskar, *Introduction to Statistical Relational Learning*, MIT Press, 2007.
- [50] W. Gilks, A. Thomas, D. Spiegelhalter, A language and program for complex Bayesian modelling, *The Statistician* 43 (1993) 169–178.
- [51] J. Gill, Computational complexity of probabilistic Turing machines, *SIAM Journal on Computing* 6 (4) (1977) 675–695.
- [52] S. Glesner, D. Koller, Constructing flexible dynamic belief networks from first-order probabilistic knowledge bases, in: *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, 1995, pp. 217–226.
- [53] R. P. Goldman, E. Charniak, Dynamic construction of belief networks, in: *Conference of Uncertainty in Artificial Intelligence*, 1990, pp. 90–97.

- [54] J. Goldsmith, M. Hagen, M. Mundhenk, Complexity of DNF minimization and isomorphism testing for monotone formulas, *Information and Computation* 206 (6) (2008) 760–775.
- [55] E. Grädel, Finite model theory and descriptive complexity, in: *Finite Model Theory and its Applications*, Springer, 2007, pp. 125–229.
- [56] E. Gribkoff, G. Van den Broeck, D. Suciu, Understanding the complexity of lifted inference and asymmetric weighted model counting, in: *Conference on Uncertainty in Artificial Intelligence*, 2014.
- [57] A. Grove, J. Halpern, D. Koller, Asymptotic conditional probabilities: the unary case, *SIAM Journal on Computing* 25 (1) (1996) 1–51.
- [58] P. Haddawy, Generating Bayesian networks from probability logic knowledge, in: *Conference on Uncertainty in Artificial Intelligence*, 1994, pp. 262–269.
- [59] D. Heckerman, An empirical comparison of three inference methods, in: R. D. Shachter, L. N. Kanal, J. F. Lemmer (eds.), *Uncertainty in Artificial Intelligence 4*, Elsevier Science Publishers, North-Holland, 1990, pp. 283–303.
- [60] D. Heckerman, C. Meek, D. Koller, Probabilistic entity-relationship models, PRMs, and plate models, in: L. Getoor, B. Taskar (eds.), *Introduction to Statistical Relational Learning*, MIT Press, 2007, pp. 201–238.
- [61] M. C. Horsch, D. Poole, A dynamic approach to probabilistic inference using Bayesian networks, in: *Conference of Uncertainty in Artificial Intelligence*, 1990, pp. 155–161.
- [62] M. Jaeger, Relational Bayesian networks, in: D. Geiger, P. P. Shenoy (eds.), *Conference on Uncertainty in Artificial Intelligence*, Morgan Kaufmann, San Francisco, California, 1997, pp. 266–273.
- [63] M. Jaeger, Complex probabilistic modeling with recursive relational Bayesian networks, *Annals of Mathematics and Artificial Intelligence* 32 (2001) 179–220.
- [64] M. Jaeger, Probabilistic role models and the guarded fragment, in: *Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU)*, 2004, pp. 235–242.
- [65] M. Jaeger, Lower complexity bounds for lifted inference, *Theory and Practice of Logic Programming* 15 (2) (2014) 246–264.
- [66] M. Jaeger, G. Van Den Broeck, Liftability of probabilistic inference: Upper and lower bounds, in: *2nd Statistical Relational AI (StaRAI-12) Workshop*, 2012.
- [67] S. M. Kazemi, A. Kimmig, G. Van den Broeck, D. Poole, New liftable classes for first-order probabilistic inference, in: *NIPS*, 2016.
- [68] K. Kersting, Lifted probabilistic inference, in: L. D. Raedt, C. Bessiere, D. Dubois, P. Doherty, P. Frasconi, F. Heintz, P. Lucas (eds.), *European Conference on Artificial Intelligence*, IOS Press, 2012, (Invited Talk at the Frontiers of AI Track).
- [69] K. Kersting, L. D. Raedt, S. Kramer, Interpreting Bayesian logic programs, in: *AAAI-2000 Workshop on Learning Statistical Models from Relational Data*, 2000.
- [70] C. Koch, MayBMS: A system for managing large uncertain and probabilistic databases, in: C. C. Aggarwal (ed.), *Managing and Mining Uncertain Data*, Springer-Verlag, 2009, pp. 149–184.
- [71] D. Koller, N. Friedman, *Probabilistic Graphical Models: Principles and Techniques*, MIT Press, 2009.

- [72] D. Koller, A. Y. Levy, A. Pfeffer, P-CLASSIC: A tractable probabilistic description logic, in: AAAI, 1997, pp. 390–397.
- [73] D. Koller, A. Pfeffer, Object-oriented Bayesian networks, in: Conference on Uncertainty in Artificial Intelligence, 1997, pp. 302–313.
- [74] D. Koller, A. Pfeffer, Probabilistic frame-based systems, in: National Conference on Artificial Intelligence (AAAI), 1998, pp. 580–587.
- [75] J. Kwisthout, The computational complexity of probabilistic inference, Tech. Rep. ICIS–R11003, Radboud University Nijmegen, The Netherlands (2011).
- [76] J. H. P. Kwisthout, H. L. Bodlaender, L., V. der Gaag, The necessity of bounded treewidth for efficient inference in Bayesian networks, in: European Conference on Artificial Intelligence, 2010, pp. 237–242.
- [77] R. E. Ladner, Polynomial space counting problems, SIAM Journal of Computing 18 (6) (1989) 1087–1097.
- [78] K. B. Laskey, MEBN: A language for first-order Bayesian knowledge bases, Artificial Intelligence 172 (2-3) (2008) 140–178.
- [79] H. R. Lewis, Complexity results for classes of quantificational formulas, Journal of Computer and System Sciences 21 (1980) 317–353.
- [80] L. Libkin, Elements of Finite Model Theory, Springer, 2004.
- [81] C. Lin, J. Liu, P. Lu, A simple FPTAS for counting edge covers, in: ACM-SIAM Symposium on Discrete Algorithms, 2014, pp. 341–348.
- [82] J. Liu, P. Lu, C. Zhang, FPTAS for counting weighted edge covers, in: European Symposium on Algorithms, 2014, pp. 654–665.
- [83] T. Lukasiewicz, U. Straccia, Managing uncertainty and vagueness in description logics for the semantic web, Journal of Web Semantics 6 (2008) 291–308.
- [84] D. Lunn, C. Jackson, N. Best, A. Thomas, D. Spiegelhalter, The BUGS Book: A Practical Introduction to Bayesian Analysis, CRC Press/Chapman and Hall, 2012.
- [85] D. Lunn, D. Spiegelhalter, A. Thomas, N. Best, The BUGS project: Evolution, critique and future directions, Statistics in Medicine 28 (2009) 3049–3067.
- [86] S. Mahoney, K. B. Laskey, Network engineering for complex belief networks, in: Conference on Uncertainty in Artificial Intelligence, 1996.
- [87] V. Mansinghka, A. Radul, CoreVenture: a highlevel, reflective machine language for probabilistic programming, in: NIPS Workshop on Probabilistic Programming, 2014.
- [88] D. D. Mauá, F. G. Cozman, The effect of combination functions on the complexity of relational Bayesian networks, in: Conference on Probabilistic Graphical Models — JMLR Workshop and Conference Proceedings, vol. 52, 2016, p. 333.
- [89] B. Milch, B. Marthi, D. Sontag, S. Russell, D. L. Ong, A. Kolobov, BLOG: Probabilistic models with unknown objects, in: IJCAI, 2005.
- [90] B. Milch, L. S. Zettlemoyer, K. Kersting, M. Haimes, L. P. Kaelbling, Lifted probabilistic inference with counting formulas, in: AAAI, 2008, pp. 1062–1068.

- [91] T. Mitchell, W. Cohen, E. Hruschka, P. Talukdar, J. Betteridge, A. Carlson, B. Dalvi, M. Gardner, B. Kisiel, J. Krishnamurthy, N. Lao, K. Mazaitis, T. Mohamed, N. Nakashole, E. Platanios, A. Ritter, M. Samadi, B. Settles, R. Wang, D. Wijaya, A. Gupta, X. Chen, A. Saparov, M. Greaves, J. Welling, Never-ending learning, in: Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence (AAAI-15), 2015.
- [92] R. E. Neapolitan, Learning Bayesian Networks, Prentice Hall, 2003.
- [93] L. Ngo, P. Haddawy, Answering queries from context-sensitive probabilistic knowledge bases, Theoretical Computer Science 171 (1–2) (1997) 147–177.
- [94] M. Niepert, G. Van den Broeck, Tractability through exchangeability: a new perspective on efficient probabilistic inference, in: AAAI Conference on Artificial Intelligence, 2014, pp. 2467–2475.
- [95] M. Ogiwara, A complexity theory for feasible closure properties, Journal of Computer and System Sciences 46 (1993) 295–325.
- [96] H. Palacios, B. Bonet, A. Darwiche, H. Geffner, Pruning conformant plans by counting models on compiled d-DNNF representations, in: International Conference on Automated Planning and Scheduling, 2005, pp. 141–150.
- [97] C. H. Papadimitriou, A note on succinct representations of graphs, Information and Control 71 (1986) 181–185.
- [98] C. H. Papadimitriou, Computational Complexity, Addison-Wesley Publishing, 1994.
- [99] J. Pearl, Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference, Morgan Kaufmann, San Mateo, California, 1988.
- [100] J. Pearl, Causality: models, reasoning, and inference, Cambridge University Press, Cambridge, United Kingdom, 2000.
- [101] J. Pearl, Causality: models, reasoning, and inference (2nd edition), Cambridge University Press, Cambridge, United Kingdom, 2009.
- [102] A. Pfeffer, IBAL: a probabilistic rational programming language, in: International Joint Conference on Artificial Intelligence, 2001, pp. 733–740.
- [103] D. Poole, Probabilistic Horn abduction and Bayesian networks, Artificial Intelligence 64 (1993) 81–129.
- [104] D. Poole, The independent choice logic for modelling multiple agents under uncertainty, Artificial Intelligence 94 (1/2) (1997) 7–56.
- [105] D. Poole, First-order probabilistic inference, in: International Joint Conference on Artificial Intelligence (IJCAI), 2003, pp. 985–991.
- [106] D. Poole, The Independent Choice Logic and beyond, in: L. D. Raedt, P. Frasconi, K. Kersting, S. Muggleton (eds.), Probabilistic Inductive Logic Programming, vol. 4911 of Lecture Notes in Computer Science, Springer, 2008, pp. 222–243.
- [107] D. Poole, Probabilistic programming languages: Independent choices and deterministic systems, in: R. Dechter, H. Geffner, J. Y. Halpern (eds.), Heuristics, Probability and Causality — A Tribute to Judea Pearl, College Publications, 2010, pp. 253–269.
- [108] H. Poon, P. Domingos, Sum-product networks: a new deep architecture, in: Conference on Uncertainty in Artificial Intelligence, 2011.

- [109] O. Pourret, P. Naim, B. Marcot, *Bayesian Networks — A Practical Guide to Applications*, Wiley, 2008.
- [110] J. S. Provan, M. O. Ball, The complexity of counting cuts and of computing the probability that a graph is connected, *SIAM Journal on Computing* 12 (4) (1983) 777–788.
- [111] L. D. Raedt, *Logical and Relational Learning*, Springer, 2008.
- [112] L. D. Raedt, K. Kersting, Probabilistic inductive logic programming, in: *International Conference on Algorithmic Learning Theory*, 2004, pp. 19–36.
- [113] R. Ramachandran, G. Qi, K. Wang, J. Wang, J. Thornton, Probabilistic reasoning in DL-Lite, in: *Pacific Rim International Conference on Trends in Artificial Intelligence*, 2012, pp. 480–491.
- [114] F. Riguzzi, The distribution semantics is well-defined for all normal programs, in: F. Riguzzi, J. Vennekens (eds.), *International Workshop on Probabilistic Logic Programming*, vol. 1413 of *CEUR Workshop Proceedings*, 2015, pp. 69–84.
- [115] D. Roth, On the hardness of approximate reasoning, *Artificial Intelligence* 82 (1-2) (1996) 273–302.
- [116] S. Sanner, D. McAllester, Affine algebraic decision diagrams (AADDs) and their application to structured probabilistic inference, in: *International Joint Conference on Artificial Intelligence*, 2005, pp. 1384–1390.
- [117] T. Sato, A statistical learning method for logic programs with distribution semantics, in: *Int. Conference on Logic Programming*, 1995, pp. 715–729.
- [118] T. Sato, Y. Kameya, Parameter learning of logic programs for symbolic-statistical modeling, *Journal of Artificial Intelligence Research* 15 (2001) 391–454.
- [119] J. Simon, On some central problems in computational complexity, Tech. Rep. TR75-224, Department of Computer Science, Cornell University (1975).
- [120] S. Singh, C. Mayfield, S. Mittal, S. Prabhakar, S. Hambrusch, R. Shah, Orion 2.0: Native support for uncertain data, in: *SIGMOD*, 2008, pp. 1239–1241.
- [121] T. Sommestad, M. Ekstedt, P. Johnson, A probabilistic relational model for security risk analysis, *Computers and Security* 29 (2010) 659–679.
- [122] D. Sontag, D. Roy, Complexity of inference in latent Dirichlet allocation, in: *Advances in Neural Information Processing Systems*, vol. 24, 2011, pp. 1008–1016.
- [123] D. Suciu, D. Oiteanu, C. Ré, C. Koch, *Probabilistic Databases*, Morgan & Claypool Publishers, 2011.
- [124] N. Taghipour, D. Fierens, G. Van den Broeck, J. Davis, H. Blockeel, Completeness results for lifted variable elimination, in: *Proceedings of the International Conference on Artificial Intelligence and Statistics (AISTATS)*, Scottsdale, USA, 2013, pp. 572–580.
- [125] M. Tenorth, M. Beetz, KnowRob: A knowledge processing infrastructure for cognition-enabled robots, *The International Journal of Robotics Research* 32 (5) (2013) 566–590.
- [126] S. Tobies, The complexity of reasoning with cardinality restrictions and nominals in expressive description logics, *Journal of Artificial Intelligence Research* 12 (2000) 199–217.
- [127] S. Toda, PP is as hard as the polynomial-time hierarchy, *SIAM Journal of Computing* 20 (5) (1991) 865–877.



- [128] S. Toda, O. Watanabe, Polynomial-time 1-Turing reductions from  $\#PH$  to  $\#P$ , *Theoretical Computer Science* 100 (1992) 205–221.
- [129] S. P. Vadhan, The complexity of counting in sparse, regular and planar graphs, *SIAM Journal of Computing* 31 (2) (2001) 398–427.
- [130] L. G. Valiant, The complexity of computing the permanent, *Theoretical Computer Science* 8 (1979) 189–201.
- [131] L. G. Valiant, The complexity of enumeration and reliability problems, *SIAM Journal of Computing* 8 (3) (1979) 410–421.
- [132] L. G. Valiant, Accidental algorithms, in: *Annual IEEE Symposium on Foundations of Computer Science*, 2006, pp. 509–517.
- [133] G. Van den Broeck, On the completeness of first-order knowledge compilation for lifted probabilistic inference, in: *Neural Processing Information Systems*, 2011, pp. 1386–1394.
- [134] G. Van den Broeck, J. Davis, Conditioning in first-order knowledge compilation and lifted probabilistic inference, in: *AAAI Conference on Artificial Intelligence*, 2012, pp. 1961–1967.
- [135] G. Van den Broeck, M. Wannes, A. Darwiche, Skolemization for weighted first-order model counting, in: *International Conference on Principles of Knowledge Representation and Reasoning*, 2014.
- [136] M. Y. Vardi, The complexity of relational query languages, in: *Annual ACM Symposium on Theory of Computing*, 1982, pp. 137–146.
- [137] D. Z. Wang, E. Michelaks, M. Garofalakis, J. M. Hellerstein, BAYESSTORE: Managing large, uncertain data repositories with probabilistic graphical models, in: *VLDB Endowment*, vol. 1, 2008, pp. 340–351.
- [138] M. P. Wellman, J. S. Breese, R. P. Goldman, From knowledge bases to decision models, *Knowledge Engineering Review* 7 (1) (1992) 35–53.
- [139] J. Widom, Trio: A system for data, uncertainty, and lineage, in: C. C. Aggarwal (ed.), *Managing and Mining Uncertain Data*, Springer-Verlag, 2009, pp. 113–148.
- [140] M. Xia, W. Zhao,  $\#3$ -regular bipartite planar vertex cover is  $\#P$ -complete, in: *Theory and Applications of Models of Computation*, vol. 3959, 2006, pp. 356–364.